

# CONFIGURATIONS OF AN ARTICULATED ARM AND SINGULARITIES OF SPECIAL MULTI-FLAGS

FERNAND PELLETIER<sup>1</sup> & MAYADA SLAYMAN<sup>2</sup>

**ABSTRACT.** On one hand P. Mormul has classified the singularities of special multi-flags in terms of "EKR class" encoding by sequences  $j_1 \cdots j_k$  of integers (cf [7] and [8]). On the other hand, A-L. Castro and R. Montgomery in [4], have proposed a codification of singularities of multi-flags **RC** and **RVT** codes. The most important results of this paper describe a decomposition of each "EKR" class of depth at most 1 and, for  $k \leq 4$ , at most 2 in terms of class in **RVT** codes, and give an interpretation of such "EKR" classes and **RVT** classes in terms of configurations of an articulated arm.

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## 1. INTRODUCTION AND RESULTS

A **special multi-flag** of step  $m \geq 1$  and length  $k \geq 1$  is a sequence (see [7]):

$$\mathcal{F} : D = D_k \subset D_{k-1} \subset \cdots \subset D_j \subset \cdots \subset D_1 \subset D_0 = TM$$

of distributions of constant rank on a manifold  $M$  of dimension  $(k+1)m+1$  which satisfies the following conditions:

- (i)  $D_{j-1} = [D_j, D_j]$  is the distribution generated by all Lie bracket of sections of  $D_j$ .
- (ii)  $D_j$  is a distribution of constant rank  $(k-j+1)m+1$ .
- (iii) Each Cauchy characteristic subdistribution<sup>3</sup>  $L(D_j)$  of  $D_j$  is a subdistribution of constant corank one in each  $D_{j+1}$ , for  $j = 1, \dots, k-1$ , and  $L(D_k) = 0$ .
- (iv) there exists a completely integrable subdistribution  $F \subset D_1$  of corank one in  $D_1$ .

(see section 2.1 for a more precise definition)

The notion of special multi-flags is described in some ways in [13] and [8]. Furthermore, for  $m \geq 2$ , it is proved in [2] and [16] that the existence of a completely integrable subdistribution  $F$  of  $D_1$  implies property (iii), and when such a distribution  $F$  exists, it is than unique ( see Theorem 2.1). When  $m = 1$  a special multi-flag is a Goursat flag, and, in this case, the conditions (iii) and (iv) are automatically satisfied but for such a distribution  $F$  is not unique. One fundamental result on Goursat flags is the existence of locally universal Goursat distributions proved by R. Montgomery and M. Zhitomirskii in [10]: the "monster Goursat manifold" which is constructed by applying Cartan prolongations  $k$  times. On the other hand, the kinematic system of a car with  $k-1$  trailers can be described by an appropriate Goursat distribution  $\Delta_k$  on  $\mathbb{R}^2 \times (\mathbb{S}^1)^k$  and moreover, this configuration space is diffeomorphic to the Cartan prolongation of the distribution  $\Delta_{k-1}$  on  $\mathbb{R}^2 \times (\mathbb{S}^1)^{k-1}$  (see Appendix D of [10]).

*Theorem 1 of this paper (which is announced in [12]) is a generalization of this last result for special multi-flags of step  $m \geq 2$ . As all the arguments used to show this results are basic arguments for the poof of Theorem 2 and Theorem 3 of this paper, we give here a complete this result*

More precisely, a special multi-flag can be considered as a generalization of the notion of Goursat flags and the fundamental result of [2] and [16] is again obtained by Cartan prolongation (see also [8]). So, in this situation, we can also defined a "monster tower" by successive Cartan prolongations of  $T\mathbb{R}^{m+1}$  (see for instance [2], [16], [4] or [5]). On the other hand, we can construct a kinematic system, called articulated arm in [15], and also called system of rigid bars in [6]. The configuration space  $\mathcal{C}^k(m)$  of such a kinematic system is diffeomorphic to  $\mathbb{R}^{m+1} \times (\mathbb{S}^m)^k$ , and and this system is characterized by a distribution  $\mathcal{D}_k$  which generates a special multi-flags of length  $k$  (see section 3.1).

<sup>1</sup>Université de Savoie, Laboratoire de Mathématiques (LAMA) Campus Scientifique, 73376 Le Bourget-du-Lac Cedex, France. E-Mail: pelletier@univ-savoie.fr.

<sup>2</sup>Lebanese University, Mathematics Department, Faculty of Sciences, Lebanon. E-Mail: mslyman@ul.edu.lb

<sup>3</sup>see section 2.1

On one hand, by Cartan prolongations, we have a tower of projective bundles: (see section 2.2)

$$(1) \quad \cdots \rightarrow P^k(m) \rightarrow P^{k-1}(m) \rightarrow \cdots \rightarrow P^1(m) \rightarrow P^0(m) := \mathbb{R}^{m+1}$$

Each manifold  $P^j(m)$  is provided with a distribution  $\Delta_j$  which is the Cartan prolongation of  $\Delta_{j-1}$

On the other hand we can also define a natural notion of "spherical prolongation" which also gives rise to a tower of sphere bundles (see section 2.3)

$$(2) \quad \cdots \rightarrow \hat{P}^k(m) \rightarrow \hat{P}^{k-1}(m) \rightarrow \cdots \rightarrow \hat{P}^1(m) \rightarrow \hat{P}^0(m) := \mathbb{R}^{m+1}$$

Again, each manifold  $\hat{P}^j(m)$  is provided with a distribution  $\hat{\Delta}_j$  which is the spherical prolongation of  $\hat{\Delta}_{j-1}$ .

Note that we have a canonical 2-fold covering

$$\hat{P}^k(m) \rightarrow P^k(m)$$

for any  $k \geq 1$  and  $m \geq 2$ .

In this context, we have the following result announced in [12]:

**Theorem 1.**

Let be  $\hat{\Delta}_k$  the canonical distribution obtained on  $\hat{P}^k(m)$  after  $k$ -fold "spherical prolongation". Then we have:

For each  $k \geq 1$  and  $m \geq 2$ , there exists a diffeomorphism  $F^k$  from  $\hat{P}^k(m)$  on  $C^k(m)$  such that:

(i) if  $\hat{\pi}^k : \hat{P}^k(m) \rightarrow \hat{P}^{k-1}(m)$  and  $\rho^k : C^k(m) \rightarrow C^{k-1}(m)$  are the canonical projections, we have:

$$\rho^k \circ F^k = F^{k-1} \circ \hat{\pi}^k$$

(ii)  $F_*^k(\hat{\Delta}_k) = \mathcal{D}_k$

In particular, this result gives a positive answer to a conjecture proposed in section 6 of [5].

The singularities of special multi-flags were firstly described by P Mormul in [7] and [8]. This classification was founded on one hand, on a generalization of Cartan prolongation and, on the other hand, on some "operation" denoted  $\mathbf{j}$  which produce a new  $(m+1)$ -distributions from the older ones. So P. Mormul constructs a coding system for labeling singularity classes of germs of special multi-flags which he called "Extended Kumpera-Ruiz classes" and "EKR classes" in short (for more details see section 5.1). An EKR class is coded by a sequence  $j_1 \cdots j_k$  so that  $j_{l+1} \leq 1 + \sup\{j_1, \dots, j_l\}$ . The integer  $\sup\{j_1, \dots, j_k\} - 1$  is called the depth of the EKR class.

More recently, in [4], A-L. Castro and R. Montgomery have proposed a codification of singularities of multi-flags founded on the tower of projective bundles (1). They use **RC** and **RVT** codes. So they get the classification of singularities of special multi-flags in terms of **RC** or **RVT** classes

*The essential results of this paper is to give a decomposition of EKR class of depth 1 in terms of **RVT** classes and also to give an interpretation of EKR class of depth 1 and **RVT** classes in terms of configurations of an articulated arm*

More precisely, in the tower (1) we can define sub-towers by taking the tower of Cartan prolongation of any fiber of  $P^j(m) \rightarrow P^{j-1}(m)$ . We get the so called "baby monsters" in [4], so that, in each vector space  $\Delta_k(p) \subset T_p P^k(m)$  we have a family of "critical" hyperplanes coming from these "baby monsters". One of them is the vertical space  $V_p P^k(m)$  i.e. the tangent space to a fiber of  $P^k(m) \rightarrow P^{k-1}(m)$ . A point  $p \in P^k(m)$  can be written  $p = (p_{k-1}, z)$  where  $p_{k-1} \in P^{k-1}(m)$  and  $z$  is a line in  $\Delta_{k-1}(p_{k-1})$ . The point  $p$  is called *vertical* if  $z$  is tangent to the fiber at  $p_{k-1}$ , *tangency* if  $z$  is not vertical but belongs to one critical hyperplane and otherwise  $p$  is called *regular*. So we can affect to  $p$  a word in letters  $\{R, V, T\}$  so that the letters of rank  $l$  is  $R$  or  $V$  or  $T$  according to the fact that the projection of  $p$  onto  $P^l(m)$  is regular or vertical or tangency respectively.

In a word  $\omega$ , a sub-word of type  $R^h$  or  $T^l$  means a sequence of  $h$  (resp.  $l$ ) consecutive letters  $R$  (resp.  $T$ ) if  $h > 0$  (resp.  $l > 0$ ), and no letter  $R$  (resp.  $T$ ) if  $h = 0$  (resp.  $l = 0$ ). For any EKR class of 1-depth we will denote by  $\{i_1, \dots, i_\nu\}$  the set  $\{i \text{ such that } j_i = 2\}$ . The following result gives a complete description of EKR classes of depth 1 in terms of class of word in **RVT** codes.

**Theorem 2.**

(1) Each EKR class  $\Sigma_{j_1 \dots j_k}$  of depth 1 is an analytic manifold of codimension  $\nu$ .

- (2) Denote by  $\mathcal{C}_{R^{h_0}VR^{h_1}\dots VR^{h_\nu}}$  a class of point whose **RVT** code is  $R^{h_0}VR^{h_1}\dots VR^{h_\nu}$ . Then any class  $\mathcal{C}_\omega$  of point  $p$  whose **RVT** code is  $\omega$  is contained in an EKR class  $\Sigma_{j_1\dots j_k}$  if and only if  $\omega$  is of type  $R^{h_0}VR^{h_1}\dots VR^{h_\nu}$  where each letter  $V$  is exactly at rank  $i_1, \dots, i_\nu$ . Such a class  $\mathcal{C}_{R^{h_0}VR^{h_1}\dots VR^{h_\nu}}$  is an analytic submanifold of  $\Sigma_{j_1\dots j_k}$  of codimension  $l_1 + \dots + l_\nu$
- (3) The EKR class  $\Sigma_{j_1\dots j_k}$  is the union of all class  $\mathcal{C}_{R^{h_0}VR^{h_1}\dots VR^{h_\nu}}$  which satisfy property (2).

**Theorem 3.**

Let be  $(M_0, \dots, M_k)$  an articulated arm in  $\mathbb{R}^{m+1}$

- (1) a configuration  $q \in \mathcal{C}^k(m)$  of the articulated arm belongs to the EKR class  $\Sigma_{j_1\dots j_k}$  of depth 1 if and only if, for this configuration, the segments  $[M_{i-2}, M_{i-1}]$  and  $[M_{i-1}, M_i]$  are orthogonal in  $M_{i-1}$  for all  $i = i_1, \dots, i_\nu$
- (2) There exists a family  $\{K_0^\lambda, \dots, K_{\kappa_\lambda}^\lambda\}_{\lambda=1, \dots, \nu}$ , with  $\kappa_\lambda = i_{\lambda+1} - i_\lambda - 1$  for  $\lambda = 1, \dots, \nu - 1$  and  $\kappa_\nu = k - i_\nu - 1$  of directions in  $\mathbb{R}^{m+1}$  such that  $q$  belongs to the class  $\mathcal{C}_{R^{h_0}VT^{l_1}R^{h_1}\dots VT^{l_\nu}R^{h_\nu}} \subset \Sigma_{j_1\dots j_k}$  if and only if
- each segment  $[M_{i_\lambda+l-1}, M_{i_\lambda+l}]$  is orthogonal to  $K_l^\lambda$  in  $M_{i_\lambda+l-1}$ , for  $l = 0, \dots, l_\lambda$  and  $\lambda = 1, \dots, \nu$ ;
  - each pair of consecutive segments  $[M_{i-2}, M_{i-1}]$  and  $[M_{i-1}, M_i]$  are not orthogonal in  $M_{i-1}$  for all  $i$  which do not belongs to the union of the previous sets of index.

We finally give the same type of results for EKR classes of depth at most 2 for  $1 \leq k \leq 4$  in the last section (section 5.4).

This paper is self-contained and organized as follows.

In Section 2 we recall at first all the context and essential results about special multi-flags which are used in this paper. In a second subsection, We present a summary on Cartan prolongation and tower of projective bundle. Spherical prolongations, tower of sphere bundles and their properties are developed in the last subsection.

Section 3 is devoted to the context of the configurations of an articulated arm of length  $k \geq 1$  in  $\mathbb{R}^{m+1}$ . The space  $\mathcal{C}^k(m)$  of such configuration is presented in the first subsection. The relation between tower of sphere bundles and  $\mathcal{C}^k(m)$  is given in the second subsection in which the reader can find of proof of Theorem 1. Finally we present the hyperspherical coordinates on  $\mathcal{C}^k(m)$  in the last subsection.

In the first subsection of section 4, according to [4], we present a summary on the **RC** and **RVT** codes and adapt these codes to the context of tower of sphere bundles. The following subsection gives some interpretations of the property of *verticality* in terms of configurations of an articulated arm. In the same way, some interpretations of the property of *tangency* are given in the last subsection.

Section 5 is devoted to the relation between EKR classes of depth at most 1 and **RVT** classes. In the first subsection, we summary the definition and procedures about EKR classes according to [7] and [8]. The following subsection gives a global description of EKR classes in terms of **RVT** classes. In particular, it contains a proof of theorem 2. The following subsection presents an interpretation in terms of configurations of an articulated arm for EKR classes (of depth at most 1) and **RVT** classes. In particular it contains a proof of Theorem 3. Finally, the last subsection develops, for  $1 \leq k \leq 4$ , the decomposition of EKR classes of depth at most 2 in **RVT** classes and the corresponding interpretation in terms of configurations of an articulated arm.

We end this paper by some commentaries about these results and the results contained in [4], [5] and [9].

## 2. PRELIMINARIES

### 2.1. Special multi-flags.

A distribution  $D$  on a manifold  $M$  is an assignment  $D : x \rightarrow D_x \subset TM$  of subspace  $D_x$  of the tangent space  $T_x M$ . A local vector field  $X$  on  $M$  is tangent to  $D$  if for any  $X(x)$  belongs to  $D_x$  for all  $x$  in the open set on which  $X$  is defined. A distribution is called a smooth distribution if there exists a set  $\mathcal{X}$  of local vector fields such that  $D_x$  is generated by the set  $\{X(x), X \in \mathcal{X}\}$ , we then say that  $D$  is generated by  $\mathcal{X}$ .

In this paper any distribution will be smooth and we denote by  $\Gamma(D)$  the set of all local vector fields which are tangent to  $D$ . Such a distribution will be called a distribution of constant rank if  $D$  defines a subbundle of  $TM$ . According to [2] and [16], any pair  $(M, D)$  of a distribution of constant rank on a smooth manifold  $M$  will be called a **differential system**. Given two differential systems  $(M, D)$  and  $(N, \Delta)$ , we will say that  $(M, D, x)$  is **locally equivalent** to  $(N, \Delta, y)$  if there exists an open neighbourhood  $U$  of  $x$  in  $M$  and a diffeomorphism  $\phi$  from  $U$  onto an open neighbourhood  $V$  of  $y = \phi(x)$  in  $N$  so that  $\phi_*(D|_U) = \Delta|_V$ .

The **Lie square** of a distribution  $D$  is the distribution denoted  $D^2$  which is generated by the sets  $\Gamma(D)$  and  $\{[X, Y], X, Y \in \Gamma(D)\}$ . The **Cauchy characteristic distribution**  $L(D)$  of a distribution  $D$  is the distribution generated by the set vector fields  $\{[X, Y], X, Y \in \Gamma(D) \text{ such that } [X, Y](x) \in D_x\}$ . If  $L(D)$  is a distribution of constant rank, then it is an integrable distribution.

A **special multi-flag** of step  $m \geq 2$  and length  $k \geq 1$  is a sequence (see [7]):

$$\mathcal{F} : D = D_k \subset D_{k-1} \subset \cdots \subset D_j \subset \cdots \subset D_1 \subset D_0 = TM$$

of distributions of constant rank on a manifold  $M$  of dimension  $(k+1)m+1$  which satisfies the following conditions:

- (i)  $D_{j-1} = (D_j)^2$ .
- (ii)  $D_j$  is a distribution of constant rank  $(k-j+1)m+1$ .
- (iii) Each Cauchy characteristic subdistribution  $L(D_j)$  of  $D_j$  is a subdistribution of constant corank one in each  $D_{j+1}$ , for  $j = 1, \dots, k-1$ , and  $L(D_k) = 0$ .
- (iv) there exists a completely integrable subdistribution  $F \subset D_1$  of corank one in  $D_1$ .

In the following, a flag  $\mathcal{F}$  which satisfies conditions (i), (ii) but not conditions (iii) and (iv) will be just called a **multi-flag** of step  $m$  or a **m-flag** and we say that  $\mathcal{F}$  is generated by  $D$ .

The necessary and sufficient condition of a multi-flag to be a special multi-flag is given by the following result (see [2] Proposition 1.3 and [16] Theorem 6.2)

**Theorem 2.1.** [2],[16] for  $k \geq 2$ , a  $m \geq 1$  consider a multi-flag of step  $m$  :

$$\mathcal{F} : D = D_k \subset D_{k-1} \subset \cdots \subset D_j \subset \cdots \subset D_1 \subset D_0 = TM$$

$\mathcal{F}$  is a special multi-flag if and only if there exists a completely integrable subbundle  $F$  of  $D_1$  of corank 1. Moreover, if such a subbundle  $F$  exists,  $F$  is unique.

According to the previous the definition of a special multi-flag, we obtain the following sandwich flag:

$$(3) \quad \begin{array}{ccccccc} D_k & \subset & D_{k-1} & \subset \cdots \subset & D_j & \subset \cdots \subset & D_1 \subset D_0 = TM \\ \cup & & \cup & & \cup & & \cup \\ L(D_{k-1}) & \subset & L(D_{k-2}) & \subset \cdots \subset & L(D_{j-1}) & \subset \cdots \subset & F \end{array}$$

All vertical inclusions in this diagram are of codimension one, while all horizontal inclusions are of codimension  $k$ . The squares built by these inclusions can be perceived as certain sandwiches, i.e. each subdiagram" number  $j$  indexed by the upper left vertices  $D_j$ :

$$\begin{array}{ccc} D_j & \subset & D_{j-1} \\ \cup & & \cup \\ L(D_{j-1}) & \subset & L(D_{j-2}) \end{array}$$

is called *sandwich number  $j$* .

In a sandwich number  $j$ , at each point  $x \in M$ , in the  $(m+1)$  dimensional vector space  $D_{j-1}/L(D_{j-1})(x)$  we can look for the relative position of the  $m$  dimensional subspace  $L(D_{j-2})/L(D_{j-1})(x)$  and the 1 dimensional subspace  $D_j/L(D_{j-1})(x)$ :

- either  $L(D_{j-2})/L(D_{j-1})(x) \oplus D_j/L(D_{j-1})(x) = D_{j-1}/L(D_{j-1})(x)$
- or  $D_j/L(D_{j-1})(x) \subset L(D_{j-2})/L(D_{j-1})(x)$ .

We say that  $x \in M$  is a **Cartan point** if the first situation is true in each sandwich number  $j$ , for  $j = 2, \dots, k$ . Otherwise  $x$  is called a **singular point**.

## 2.2. Cartan prolongation and tower of projective bundles.

Let be  $D$  a distribution of constant rank  $m+1$  on a manifold  $M$  of dimension  $n$ . Classically the Grassmannian bundle  $G(D, 1)$  on  $M$  is the set

$$(4) \quad G(D, 1, M) := \bigcup_{x \in M} P(D(x), 1)$$

where  $P(D(x), 1)$  is the projective space of the vector space  $D(x)$ . So we have a bundle  $\pi : G(D, 1) \rightarrow M$  whose fiber  $\pi^{-1}(x)$  is diffeomorphic to the projective space  $\mathbb{R}P^m$ . The **rank one Cartan prolongation** is the distribution  $D^{(1)}$  defined in the following way: given a point  $(x, \lambda) \in G(D, 1)$  then

$$(5) \quad D_{(x, \lambda)}^{(1)} := d\pi^{-1}(\lambda) \subset T_{(x, \lambda)}G(D, 1, M)$$

where  $\lambda$  is a direction of  $D(x)$ . Then  $D^{(1)}$  is a distribution on  $G(D, 1, M)$  of constant rank  $m + 1$ . Let be  $M$  a manifold of dimension  $m + 1$ . As in [16], for any  $m \geq 2$  and  $k \geq 1$  starting with  $D = TM$ , we obtain inductively a tower of bundles:

$$(6) \quad \cdots \rightarrow P^k(M) \rightarrow P^{k-1}(M) \rightarrow \cdots \rightarrow P^1(M) \rightarrow P^0(M) := M$$

where, for any  $j = 0, \dots, k$ ,  $P^j(M)$  is a manifold of dimension  $(j + 1)m + 1$ , and on each  $P^j(M)$ ,  $\Delta_j$  is a distribution and these data are defined inductively by :

$$P^j(M) = G(\Delta_{j-1}, 1, P^{j-1}(M)) \text{ and } \Delta_j = (\Delta_{j-1})^{(1)} \text{ for } j = 1, \dots, k \text{ and } \Delta_0 = TM.$$

In the particular case of  $M = \mathbb{R}^{m+1}$ , for  $j = 0, \dots, k$ , we denote simply by  $P^j(m)$  the manifold  $P^j(\mathbb{R}^{m+1})$  and we have the following tower of bundles:

$$(7) \quad \cdots \rightarrow P^k(m) \rightarrow P^{k-1}(m) \rightarrow \cdots \rightarrow P^1(m) \rightarrow P^0(m) := \mathbb{R}^{m+1}$$

We have then the following result :

**Theorem 2.2.** [16]

- (1) On  $P^k(m)$ , the distribution  $\Delta_k$  generates a special multi-flag of step  $m$  and length  $k$ .
- (2) let be  $\mathcal{F} : D = D_k \subset D_{k-1} \subset \cdots \subset D_j \subset \cdots \subset D_1 \subset D_0 = TM$  a special multi-flag of step  $m \geq 2$  and length  $k \geq 1$ . Then, for any  $x \in M$ , there exists  $y \in P^k(m)$  for which the differential system  $(P^k(m), \Delta_s, y)$  is locally equivalent to the differential system  $(M, D, x)$ .

**Remark 2.1.**

The part 2 of Theorem 2.2 can be found precisely in [16] called "Drapeau Theorem". However, according to the definition of a special multi-flag, we can easily deduce this result from the following Theorem of [8]:

**Theorem 2.3.** [8] Suppose that  $D$  is a  $(m + 1)$ -dimensional distribution on a manifold  $M^{s+m}$  such that:

- (1)  $D_1 = [D, D]$  is a  $(2m + 1)$ -dimensional distribution on  $M$ ,
- (2) there exists a 1-codimensional involutive subdistribution  $E \subset D$  that preserves  $D_1$ , that is  $[E, D_1] \subset D_1$ .

Then, locally,  $D$  is equivalent to the Cartan prolongation  $(D_1/E)^{(1)}$  of  $D_1$  reduced modulo  $E$

### 2.3. Spherical prolongation, Cartan prolongation and tower of sphere bundles.

Let be  $D$  a distribution of constant rank  $m + 1$  on a manifold  $M$  of dimension  $n$ . Choose any riemannian metric  $g$  on  $M$ , and we denote by  $S(D, M, g)$  the unit sphere bundle of  $D$  associated to the induced riemannian metric on  $D$ . So we get a bundle  $\hat{\pi} : S(D, M, g) \rightarrow M$ . On  $S(D, M, g)$ , we consider the antipodal action of  $\mathbb{Z}_2$ . Clearly, the quotient of  $S(D, M, g)$  by this action can be identified with  $G(D, 1, M)$  and the associated projection  $\tau : S(D, M, g) \rightarrow G(D, 1, M)$  is a bundle morphism over  $M$ , and also a two-fold covering. In particular  $\tau$  is a local diffeomorphism. On  $S(D, M, g)$  we consider the distribution  $D^{[1]}$  defined in the following way

$$(8) \quad D_{(x, \nu)}^{[1]} := \{v \in T_{(x, \nu)} S(D, M, g) \text{ such that } d\hat{\pi}(v) = \lambda \nu \text{ for some } \lambda \in \mathbb{R}\}$$

where  $\nu$  is a norm one vector in  $D(x)$ .

The distribution  $D^{[1]}$  is called the **rank one spherical prolongation** of  $(M, D, g)$

**Remark 2.2.**

In fact, the unit sphere bundle of  $S(D, M)$  is defined as soon as we fix some riemannian metric on  $D$ . In this case, the distribution  $D^{[1]}$  is also well defined.

**Lemma 2.1.**

- (i) we have  $\tau_* D^{[1]} = D^{(1)}$
- (ii) There exists a canonical riemannian metric  $\hat{g}$  on  $S(D, M, g)$  which is uniquely defined from the riemannian metric  $g$  on  $M$ .

*Proof.*

At first we show part (i) locally. Choose a chart domain  $U$  over which  $D$  is trivial. We choose an orthonormal frame  $\{e_0, \dots, e_m\}$  of  $D$  over  $U$ . Without loss of generality we can assume that  $D|_U \equiv \mathbb{R}^n \times \mathbb{R}^{m+1}$  so, the bundle  $S(D, M, g)|_U$  is isomorphic to  $\mathbb{R}^n \times \mathbb{S}^m$  and  $G(D, 1, M)|_U$  is isomorphic to  $\mathbb{R}^n \times \mathbb{R}P^m$ . So locally,  $\tau : \mathbb{R}^n \times \mathbb{S}^m \rightarrow \mathbb{R}^n \times \mathbb{R}P^m$  is the map  $(x, \nu) \rightarrow (x, [\nu])$  where  $[\nu]$  is the line bundle generated by  $\nu$ . From the definition of  $D_{(x, \nu)}^{[1]}$  and  $D_{\mu(x, [\nu])}^{(1)}$  we have  $\tau_*(D_{(x, \nu)}^{[1]}) = D_{\tau(x, [\nu])}^{(1)}$ . As  $\tau$  is a local diffeomorphism we get the part (i) locally. On the other hand, the map  $\hat{\alpha} : S(D, M, g) \rightarrow S(D, M, g)$  given by  $\hat{\alpha}(x, \nu) = (x, -\nu)$  is a diffeomorphism which commutes with  $\tau$ . From the definition of  $D^{[1]}$ , we get

$$\hat{\alpha}_*(D_{(x, \nu)}^{[1]}) = D_{(x, -\nu)}^{[1]}$$

This ends the proof of part (i).

For part (ii), denote by  $\bar{g}$  the canonical riemannian metric on  $TM$  associated to  $g$ . As,  $S(D, M, g)$  can be considered as a submanifold on  $TM$  we get an natural induced riemannian metric  $\hat{g}$  on  $S(D, M, g)$ .  $\square$

Let be  $g_0$  and  $g_1$  two riemannian metrics on  $M$ . We denote by  $S_i(D, M)$  the sphere bundle of  $D$  associated to the metric  $g_i$ , and  $D_i^{[1]}$  the spherical prolongation of  $(M, D, g_i)$  for  $i = 0, 1$ .

**Lemma 2.2.**

*There exists a canonical isomorphism of sphere bundle  $\psi : S_0(D, M) \rightarrow S_1(D, M)$  such that  $\psi_*(D_0^{[1]}) = D_1^{[1]}$*

*Proof.*

Let be  $D^\circ = \bigcup_{x \in M} [D_x \setminus \{0\}]$ . Then  $D^\circ$  is an open submanifold of  $D \subset TM$ . On  $D^\circ$  we consider the map  $\Psi(x, u) : D^\circ \rightarrow D^\circ$  defined by

$$\Psi(x, u) = (x, \frac{u}{[g_1(u, u)]^{1/2}}).$$

If  $\Pi : D \rightarrow M$  is the projection bundle, for any  $(x, u) \in D$ , there exists a neighbourhood  $\hat{U} = \Pi^{-1}(U) \cap D^\circ$  of  $(x, u)$  in  $D^\circ$  such that over this open,  $TD|_{\hat{U}}$  can be identified with  $\hat{U} \times T_x M \times D_x$ . Then, In this context, we have :

$$(9) \quad d\Psi(v, w) = (v, -\frac{g_1(u, w)}{2[g_1(u, u)]^{3/2}}).$$

It is easy to see that  $\Psi$  is a diffeomorphism from  $D^\circ$  into itself which commutes with  $\Pi$  and which sends  $S_0(D, M)$  to  $S_1(D, M)$ . So the restriction  $\psi$  of  $\Psi$  to  $S_0(D, M)$  is a diffeomorphism onto  $S_1(D, M)$ . Moreover, from (9),  $d\Psi$  map the the linear span  $\mathbb{R}u$  into itself, for any  $u$  in the fiber  $D_x^\circ$  over  $x$ . So we have

$$\psi_*(D_0^{[1]}) = D_1^{[1]}.$$

$\square$

Consider a differential system  $(M', D')$  and  $\phi : M \rightarrow M'$  an injective immersion such that  $\phi_*(D_x) \subset D'_{\phi(x)}$  for any  $x \in M$ . Given any riemanian metric  $g'$  on  $M'$ , we get an induced riemannian metric  $g$  on  $M$  and we can consider the associated spherical prolongation then we have:

**Lemma 2.3. :**

*in the previous context, let be  $\hat{\phi} : S(D, M, g) \rightarrow S(D', M', g')$  the map defined by*

$$\hat{\phi}(x, \nu) = (\phi(x), d_x \phi(\nu)).$$

*Then  $\hat{\phi}$  is a bundle morphism over  $\phi$  which is an injective immersion and such that*

$$(i) \quad \hat{\phi}(S(D, M, g)) = S(\phi_*(D), \phi(M), g')$$

$$(ii) \quad \hat{\phi}_*(D^{[1]}) = (\phi_*(D))^{[1]} \subset (D')^{[1]}.$$

*Moreover, if  $\phi$  is a diffeomorphism such that  $\phi_*(D) = D'$ , then  $\hat{\phi}$  is also a diffeomorphism and we have  $\hat{\phi}_*(D^{[1]}) = (D')^{[1]}$  and the riemannian metric  $\hat{\phi}_*g'$  is nothing but the canonical metric  $\hat{g}$  naturally associated to  $g$  on  $M$ .*

*Proof.*

As in Lemma, consider the map  $\hat{\phi}(x, \nu) = (\phi(x), d_x \phi(\nu))$ . From our assumptions, we get a smooth map from  $S(D, M, g)$  to  $S(D', M', g')$  and from its definition, clearly  $\hat{\phi}$  is a bundle morphism over  $\phi$ . As  $\phi$  is an injective immersion, it follows that at first  $\hat{\phi}$  is injective.

Note that the tangent space  $T_{(x, \nu)} S_x$  of the fiber  $S_x$  over  $x$  of  $S(D, M, g)$  can be identified with

$$\{v \in D_x \text{ such that } g(\nu, v) = 0\}.$$

Now any  $V \in T_{(x, \nu)} S(D, M, g)$  can be written  $V = (u, v)$  with  $u \in T_x M$  and  $v \in T_{(x, \nu)} S_x$ . So we have then:

$$(10) \quad d_{(x, \nu)} \hat{\phi}(u, v) = (d_x \phi(u), d_x \phi(v))$$

So,  $\hat{\phi}$  is an immersion, from (10).

On the other hand, as  $\phi^*g' = g$ ,  $d_x \phi$  is an isometry on its range, and then,  $d_x \phi(S_x)$  is the fiber over  $\phi(x)$  of  $S(\phi_*(D), \phi(M), g')$  and we get (i).

Let be  $\hat{\pi} : S(D, M, g) \rightarrow M$  and  $\hat{\pi}' : S'(D', M', g') \rightarrow M'$  the natural projections. We have then :

$$d\hat{\pi}' \circ d\hat{\phi} = d\phi \circ d\hat{\pi}$$

So, we get :

$$\{\hat{\phi}_*(D^{[1]})\}_{\hat{\phi}(x, \nu)} = \{d\hat{\phi}(u, v), (u, v) \in T_{(x, \nu)} S(D, M, g) \mid d\hat{\pi}(u, v) = \lambda \nu \quad \lambda \in \mathbb{R}\}$$

$$\begin{aligned}
&= \{d\hat{\phi}(u, v), (u, v) \in T_{(x, \nu)}S(D, M, g) \mid d\phi \circ d\hat{\pi}(u) = d\hat{\pi}' \circ d\hat{\phi}(u, v) = \lambda d\phi(\nu) \mid \lambda \in \mathbb{R}\} \\
&= \{(\phi_*(D))^{[1]}\}_{\hat{\phi}(x, \nu)}
\end{aligned}$$

this ends the proof of (ii).

Assume now that  $\phi$  is a diffeomorphism such that  $\phi_*(D) = D'$  and let be  $\psi = \phi^{-1}$ . From the definition of  $\hat{\phi}$  and  $\hat{\psi}$ , it follows trivially that  $\hat{\psi} \circ \hat{\phi} = Id$ . On the other hand, from the definition of  $[\phi_*(D)]^{[1]}$ , as  $d_x\phi$  is an isomorphism, we must have  $\{(\phi_*(D))^{[1]}\}_{\hat{\phi}(x, \nu)} = \{(D')^{[1]}\}_{\hat{\phi}(x, \nu)}$ . Finally, by assumption,  $\phi$  is an isometry from  $(M, g)$  to  $(M', g')$ ,  $d\phi$  is also an isometry for  $(TM, \bar{g})$  and  $(TM', \bar{g}')$  if  $\bar{g}$  and  $\bar{g}'$  are the canonical riemannian metric on the tangent bundle induced by  $g$  and  $g'$  respectively. As by construction  $\hat{g}$  and  $\hat{g}'$  are the restriction respective of  $\bar{g}$  and  $\bar{g}'$  to  $S(D, M, g) \subset TM$  and  $S(D', M', g') \subset TM'$  we get the last property and this ends the proof of the Lemma.  $\square$

So, as in the context of Cartan prolongation, for any  $m \geq 2$  and  $k \geq 1$  we can define, inductively, a tower of sphere bundles (for a fixed choice of the metric  $g$  on a manifold  $M$ ):

$$(11) \quad \cdots \rightarrow \hat{P}^k(M) \rightarrow \hat{P}^{k-1}(M) \rightarrow \cdots \rightarrow \hat{P}^1(M) \rightarrow \hat{P}^0(M) := M$$

where, for any  $j = 0, \dots, k$ ,  $\hat{P}^j(M)$  is a manifold of dimension  $(j+1)m+1$ , and, on each  $\hat{P}^j(M)$  we have a canonical distribution  $\hat{\Delta}_j$  and a riemannian metric  $g_j$  on  $\hat{P}^j(M)$ , all these data are defined inductively in the following way :

$$\begin{aligned}
\hat{P}^j(M) &= S(\hat{\Delta}_{j-1}, \hat{P}^{j-1}(M), g_{j-1}), \\
\hat{\Delta}_j &= (\hat{\Delta}_{j-1})^{[1]} \text{ for } j = 1, \dots, k \text{ and } \hat{\Delta}_0 = TM, \\
g_j &\text{ is the riemannian metric } \hat{g}_{j-1} \text{ on } S(\hat{\Delta}_{j-1}, \hat{P}^{j-1}(M), g_{j-1}) \text{ associated to } g_{j-1} \text{ for } j = 1, \dots, k \text{ and } g_0 = g.
\end{aligned}$$

Note that if  $g'$  is another riemannian metric on  $M$ , according to Lemma 2.2 and Lemma 2.3, by induction we get a family of diffeomorphisms  $\psi^j$  such that, if :

$$\cdots \rightarrow \hat{P}'^k(M) \rightarrow \hat{P}'^{k-1}(M) \rightarrow \cdots \rightarrow \hat{P}'^1(M) \rightarrow \hat{P}'^0(M) := M$$

is the tower of sphere bundles associated to the choice  $g'$  on  $M$  we have, for all  $j = 0, \dots, k$  :

$$\begin{aligned}
\psi^j(\hat{P}^j(M)) &= \hat{P}'^j(M) \\
\psi^j &\text{ is fiber preserving} \\
\psi_*^j(\hat{\Delta}_j) &= \hat{\Delta}'_j
\end{aligned}$$

**So the properties which characterize the tower (11) are independent of the choice of the riemannian metric  $g$  on  $M$ .**

For simplicity we write  $\hat{P}^j(m) := \hat{P}^j(\mathbb{R}^{m+1})$  for any  $j \in \mathbb{N}$ . From Theorem 2.2, and Lemma 2.1 we get the following result :

**Theorem 2.4.** *Let be*

$$(12) \quad \cdots \rightarrow \hat{P}^k(m) \rightarrow \hat{P}^{k-1}(m) \rightarrow \cdots \rightarrow \hat{P}^1(m) \rightarrow \hat{P}^0(m) := \mathbb{R}^{m+1}$$

*the tower of sphere bundles associated to the canonical metric on  $\mathbb{R}^{m+1}$ .*

(1) *we have a canonical two-fold covering  $\tau^k : \hat{P}^k(m) \rightarrow P^k(m)$  such that*

$$\tau^k(\hat{\Delta}_s) = \Delta_k.$$

(2) *On each manifold  $\hat{P}^k(m)$ , the distribution  $\hat{\Delta}_k$  generates a special multi-flag of step  $m$  and length  $k$ .*

(3) *let be  $\mathcal{F} : D = D_k \subset D_{k-1} \subset \cdots \subset D_j \subset \cdots \subset D_1 \subset D_0 = TM$  a special multi-flag of step  $m \geq 2$  and length  $k \geq 1$ . Then, for any  $x \in M$ , there exists  $y \in \hat{P}^k(m)$  for which the differential system  $(\hat{P}^k(m), \hat{\Delta}_s, y)$  is locally equivalent to the differential system  $(M, D, x)$ .*

*The tower (12) will be called the spherical tower of special multi-flags of step  $k$ .*

### 3. TOWER OF SPHERE BUNDLES ASSOCIATED TO A KINEMATIC SYSTEM

#### 3.1. A kinematic system for special multi-flags.

We locate us in the context of [6] and [15]. Consider a set of  $k$  segments  $[M_i; M_{i+1}]$ ,  $i = 0, \dots, k-1$ , in  $\mathbb{R}^{m+1}$ , with  $m \geq 2$ , keeping a constant length  $l_i = 1$  between  $M_i$  and  $M_{i+1}$ , and the articulation occurs at points  $M_i$ , for  $i = 1, \dots, k-1$ .

Such a system is called a "k-bar system" in [6] and an "articulated arm of length  $k$ " in [15]. The kinematic evolution of the extremity  $M_0$ , under the constraint that the velocity of each point  $M_i$ ,  $i = 0, \dots, k-1$ , is collinear with the segment  $[M_i, M_{i+1}]$  is completely described in terms of hyperspherical coordinate in [15] and

result of flatness and controllability for such a system are proved in [6]. We can associate to this problem a special multi-flag of step  $m \geq 2$  and length  $k \geq 1$  as explained in the following:

We can decompose  $(\mathbb{R}^{m+1})^{k+1}$ , into a product  $\mathbb{R}_0^{m+1} \times \cdots \times \mathbb{R}_i^{m+1} \times \cdots \times \mathbb{R}_k^{m+1}$ . Let  $x_i = (x_i^1, \dots, x_i^{m+1})$  be the canonical coordinates on the space  $\mathbb{R}_i^{m+1}$  which is equipped with its canonical scalar product  $\langle \cdot, \cdot \rangle$ .  $(\mathbb{R}^{m+1})^{k+1}$  is then equipped with its canonical scalar product too.

Consider an articulated arm of length  $k$  denoted by  $(M_0, \dots, M_k)$ . On  $(\mathbb{R}^{m+1})^{k+1}$ , consider the vector fields:

$$(13) \quad \mathcal{Z}_i = \sum_{r=1}^{m+1} (x_{i+1}^r - x_i^r) \frac{\partial}{\partial x_i^r} \text{ for } i = 0, \dots, k-1$$

From our previous assumptions, the kinematic evolution of the articulated arm is described by a controlled system:

$$(14) \quad \dot{q} = \sum_{i=0}^{s-1} u_i \mathcal{Z}_i + \sum_{r=1}^{m+1} u_{n+r} \frac{\partial}{\partial x_k^r}$$

with the following constraints:

$\|x_i - x_{i+1}\| = 1$  for  $i = 0 \dots k-1$  (see [6] or [15]).

Consider the map  $\Psi_i(x_0, \dots, x_k) = \|x_i - x_{i+1}\|^2 - 1$ . Then, the **configuration space**  $\mathcal{C}^k(m)$  is the set

$$(15) \quad \{(x_0, \dots, x_k), \text{ such that } \Psi_i(x_0, \dots, x_k) = 0 \text{ for } i = 0, \dots, k-1\}$$

For  $i = 0, \dots, k-1$ , the vector field:

$$(16) \quad \mathcal{N}_i = \sum_{r=1}^{m+1} (x_{i+1}^r - x_i^r) \left[ \frac{\partial}{\partial x_{i+1}^r} - \frac{\partial}{\partial x_i^r} \right]$$

is proportional to the gradient of  $\Psi_i$ .

So the tangent space  $T_q \mathcal{C}^k(m)$  is the subspace of  $T_q(\mathbb{R}^{m+1})^{k+1}$  which is orthogonal to  $\mathcal{N}_i(q)$  for  $i = 0, \dots, k-1$ .

Denote by  $\mathcal{E}_k$  the distribution generated by the vector fields

$$\{\mathcal{Z}_0, \dots, \mathcal{Z}_{k-1}, \frac{\partial}{\partial x_k^1}, \dots, \frac{\partial}{\partial x_k^{m+1}}\}.$$

**Lemma 3.1.** [15]

Let  $\mathcal{D}_k$  be the distribution on  $\mathcal{C}^k(m)$  defined by  $\Delta(q) = T_q \mathcal{C} \cap \mathcal{E}$ . Then  $\mathcal{D}_k$  is a distribution of dimension  $m+1$  generated by

$$(x_k^r - x_{k-1}^r) \left[ \sum_{i=0}^{k-1} \prod_{j=i+1}^k \mathcal{A}_j \mathcal{Z}_i \right] + \frac{\partial}{\partial x_k^r} \text{ for } r = 1 \dots m+1$$

where  $\mathcal{A}_j(q) = -\langle \mathcal{N}_j(q), \mathcal{N}_{j-1}(q) \rangle = \langle \mathcal{Z}_j(q), \mathcal{N}_{j-1}(q) \rangle$  for  $j = 1, \dots, k-1$  and  $\mathcal{A}_k = 1$ .

**Remark 3.1.** : according to notations of Lemma 3.1, we set

$$Y_k = \left[ \sum_{i=0}^{k-1} \prod_{j=i+1}^k \mathcal{A}_j \mathcal{Z}_i \right] = \left[ \sum_{i=0}^{k-2} \prod_{j=i+1}^{k-1} \mathcal{A}_j \mathcal{Z}_i \right] + \mathcal{Z}_{k-1} \text{ and } Y_{k-1} = \left[ \sum_{i=0}^{k-3} \prod_{j=i+1}^{k-2} \mathcal{A}_j \mathcal{Z}_i \right] + \mathcal{Z}_{k-2}$$

So we have  $Y_k = \left[ \sum_{r=1}^{m+1} (x_k^r - x_{k-1}^r)(x_{k-1}^r - x_{k-2}^r) \right] Y_{k-1} + \mathcal{Z}_{k-1}$ . Moreover,  $\mathcal{D}_k$  is generated by the family

$$\{(x_k^r - x_{k-1}^r) Y_k + \frac{\partial}{\partial x_k^r} \mid r = 1 \dots m+1\}$$

The properties of  $\mathcal{D}_k$  are summarized in the following result. (see [15] also [6])

**Theorem 3.1.**

On  $\mathcal{C}^k(m)$ , the distribution  $\mathcal{D}_k$  satisfies the following properties:

- (1)  $\mathcal{D}_k$  is a distribution of rank  $m+1$ .
- (2) The distribution  $\mathcal{D}_k$  is a special multi-flag on  $\mathcal{C}^k(m)$  of step  $m$  and length  $k$ .



### 3.2. Articulated arm and spherical prolongation.

To an articulated arm on  $\mathbb{R}^{m+1}$  ( $m \geq 2$ ) of length  $k \geq 1$  we can associate the following canonical tower of sphere bundles:

$$(17) \quad \mathcal{C}^k(m) \rightarrow \mathcal{C}^{k-1}(m) \rightarrow \cdots \rightarrow \mathcal{C}^1(m) \rightarrow \mathcal{C}^0(m) := \mathbb{R}^{m+1}$$

where for  $j = 1, \dots, k$ , the projection  $\mathcal{C}^j(m) \rightarrow \mathcal{C}^{j-1}(m)$  is the restriction of the canonical projection

$$\begin{aligned} \mathbb{R}_0^{m+1} \times \cdots \times \mathbb{R}_i^{m+1} \times \cdots \times \mathbb{R}_j^{m+1} &\rightarrow \mathbb{R}_0^{m+1} \times \cdots \times \mathbb{R}_i^{m+1} \times \cdots \times \mathbb{R}_{j-1}^{m+1} \\ (x_0, \dots, x_{j-1}, x_j) &\rightarrow (x_0, \dots, x_{j-1}) \end{aligned}$$

According to Theorem 3.1, and Theorem 2.4, we know that the differential system  $(\mathcal{C}^k(m), \mathcal{D}_k)$  associated to an articulated arm of length  $k$  on  $\mathbb{R}^{m+1}$  is locally isomorphic to the canonical differential system  $(\hat{P}^k(m), \hat{\Delta}_k)$  at some appropriate points. In fact, we have more: (see Theorem 1 in the introduction)

#### Theorem 3.2.

For each  $k \geq 1$  and  $m \geq 2$ , there exists a diffeomorphism  $F^k$  from  $\hat{P}^k(m)$  on  $\mathcal{C}^k(m)$  such that:

(i) if  $\hat{\pi}^k : \hat{P}^k(m) \rightarrow \hat{P}^{k-1}(m)$  and  $\rho^k : \mathcal{C}^k(m) \rightarrow \mathcal{C}^{k-1}(m)$  are the canonical projections, we have:

$$\rho^k \circ F^k = F^{k-1} \circ \hat{\pi}^k$$

(ii)  $F_*^k(\hat{\Delta}_k) = \mathcal{D}_k$

So according to Theorem 2.4 from Theorem 3.2 we have :

**Theorem 3.3.** Let be  $\mathcal{F} : D = D_k \subset D_{k-1} \subset \cdots \subset D_j \subset \cdots \subset D_1 \subset D_0 = TM$  a special multi-flag of step  $m \geq 2$  and length  $k \geq 1$ . Then, for any  $x \in M$ , there exists  $y \in \mathcal{C}^k(m)$  for which the differential system  $(\mathcal{C}^k(m), \mathcal{D}_k, y)$  is locally equivalent to the differential system  $(M, D, x)$ .

**The end of this subsection is devoted to the proof of Theorem 3.2.** Before proving these results, we need some auxiliary results.

#### Lemma 3.2.

For  $k \geq 1$ , consider, on  $\mathcal{C}^k(m)$ , the natural decomposition:

$$[T(\mathbb{R}^{m+1})^{k+1}]|_{\mathcal{C}^k(m)} = T\mathcal{C}^k(m) \oplus [T\mathcal{C}^k(m)]^\perp$$

where  $[T\mathcal{C}^k(m)]^\perp$  is the orthogonal of  $T\mathcal{C}^k(m)$  and denote by  $\Pi_k$  the orthogonal projection of  $[T(\mathbb{R}^{m+1})^{k+1}]|_{\mathcal{C}^k(m)}$  onto  $T\mathcal{C}^k(m)$ . On the other hand, let  $\mathcal{L}_k$  be the involutive distribution whose leaves are the fibers of the natural fibration of  $\mathcal{C}^k(m)$  onto  $\mathcal{C}^{k-1}(m)$ .

(1) The family vector fields  $\{\Pi_k(\frac{\partial}{\partial x_k^r}), r = 1, \dots, m+1\}$  generates the distribution  $\mathcal{L}_k$ .

(2) The distribution  $\mathcal{D}_k$ , is generated by  $\mathcal{L}_k$  and the vector field  $X_k = \sum_{i=0}^{k-1} \prod_{j=i+1}^k \mathcal{A}_j Z_i + \mathcal{V}_k$

$$\text{where } \mathcal{V}_k = \sum_{s=1}^{m+1} (x_k^s - x_{k-1}^s) \frac{\partial}{\partial x_k^s}.$$

(3) The distribution  $\mathcal{D}_k$  is also generated by the family of vector fields

$$(x_k^r - x_{k-1}^r) X_k + \Pi_k(\frac{\partial}{\partial x_k^r})$$

*Proof.*

Let be  $\mathcal{H}_k$  the subdistribution of  $\mathcal{E}_k$  generated by the family vector fields  $\{\frac{\partial}{\partial x_k^r}, r = 1, \dots, m+1\}$ . So  $\mathcal{H}_k \cap T\mathcal{C}^k(m)$  is a distribution on  $\mathcal{C}^k(m)$  which is contained in  $\mathcal{D}_k$ . In fact, we have

$$\mathcal{L}_k = \ker d\Psi_{k-1} \cap \mathcal{H}_k = \Pi_k(\mathcal{H}_k).$$

So  $\mathcal{L}_k$  is generated by the family vector fields  $\{\Pi_k(\frac{\partial}{\partial x_k^r}), r = 1, \dots, m+1\}$ . On the other hand, as  $\mathcal{H}_k$  is the vertical bundle of the canonical projection

$$(18) \quad \begin{aligned} \mathbb{R}_0^{m+1} \times \mathbb{R}_k^{m+1} &\rightarrow \mathbb{R}_0^{m+1} \times \cdots \times \mathbb{R}_{k-1}^{m+1} \\ (x_0, \dots, x_{k-1}, x_k) &\rightarrow (x_0, \dots, x_{k-1}) \end{aligned}$$

it follows that  $\mathcal{L}_k$  is also the vertical bundle of the induced projection of  $\mathcal{C}^k(m)$  onto  $\mathcal{C}^{k-1}(m)$ . On the other hand, the fiber over  $q \in \mathcal{C}^{k-1}(m)$  of the previous fibration is the unit sphere  $S_q = \{(q, x_k) : \Psi_{k-1}(q, x_k) = 0\}$  which proves (1).

The vector field  $\mathcal{V}_k = \sum_{s=1}^{m+1} (x_k^s - x_{k-1}^s) \frac{\partial}{\partial x_k^s}$  is vertical for the projection (18) and is orthogonal to each  $S_q$ .

As  $\|\mathcal{V}_k\| = 1$  we thus have:

$$(19) \quad \Pi_k\left(\frac{\partial}{\partial x_k^r}\right) = \frac{\partial}{\partial x_k^r} - (x_k^r - x_{k-1}^r) \mathcal{V}_k$$

From Lemma 3.1,  $\mathcal{D}_k$  is generated by the family  $\{(x_k^r - x_{k-1}^r) [\sum_{i=0}^{k-1} \prod_{j=i+1}^k \mathcal{A}_j \mathcal{Z}_i] + \frac{\partial}{\partial x_k^r} \text{ for } r = 1 \cdots m+1\}$ .

So the vector field  $X_k = [\sum_{i=0}^{k-1} \prod_{j=i+1}^k \mathcal{A}_j \mathcal{Z}_i] + [\sum_{r=1}^{m+1} (x_k^r - x_{k-1}^r) \frac{\partial}{\partial x_k^r}]$  is tangent to  $\mathcal{D}_k$  but clearly this vector fields is not tangent to  $\mathcal{L}_k$ . As  $\mathcal{D}_k$  is a distribution of constant rank  $m+1$  and  $\mathcal{L}_k$  is an (integrable) subdistribution of rank  $m$ , it follows that  $\mathcal{D}_k$  is generated by  $\mathcal{L}_k$  and  $X_k$  which proves (2).

On the other hand, according to (19), each vector field  $(x_k^r - x_{k-1}^r) [\sum_{i=0}^{k-1} \prod_{j=i+1}^k \mathcal{A}_j \mathcal{Z}_i] + \frac{\partial}{\partial x_k^r}$  can be written

$$(x_k^r - x_{k-1}^r) X_k + \Pi_k\left(\frac{\partial}{\partial x_k^r}\right)$$

According to Lemma 3.1 we get (3). □

**Proposition 3.1.**

- (1) *There exists a bundle isomorphism  $\hat{\Psi} : \mathcal{D}_k \rightarrow \mathcal{C}^k(m) \times \mathbb{R}^{m+1}$*
- (2) *Let be  $\gamma_k$  a riemannian metric on the bundle  $\mathcal{D}_k$  so that the morphism  $\hat{\Psi}$  is an isometry between  $\mathcal{D}_k$  and  $\mathcal{C}^k(m) \times \mathbb{R}^{m+1}$  (for the canonical euclidian product on the fiber  $\mathbb{R}^{m+1}$ ). Then  $\hat{\Psi}$  induces a diffeomorphism  $\Psi : S(\mathcal{D}_k, \mathcal{C}^k(m), \gamma_k) \rightarrow \mathcal{C}^{k+1}(m)$  such that*
  - (i)  $\Psi$  commutes with the canonical projections  $S(\mathcal{D}_k, \mathcal{C}^k(m), \gamma_k) \rightarrow \mathcal{C}^k(m)$  and  $\mathcal{C}^{k+1}(m) \rightarrow \mathcal{C}^k(m)$ .
  - (ii)  $\Psi_*[(\mathcal{D}_k)^{[1]}] = \mathcal{D}_{k+1}$ .

*Proof.*

From Lemma 3.2 part (3), the bundle  $\mathcal{D}_k$  has  $m+1$  non-zero global sections

$$(20) \quad (x_k^r - x_{k-1}^r) X_k + \Pi_k\left(\frac{\partial}{\partial x_k^r}\right) \text{ for } r = 1 \cdots m+1$$

so  $\mathcal{D}_k$  is a trivial bundle. It follows that there exists a bundle isomorphism

$$\hat{\Psi} : \mathcal{D}_k \rightarrow \mathcal{C}^k(m) \times \mathbb{R}^{m+1} \text{ which ends part (1).}$$

Put on  $\mathcal{D}_k$  the riemannian metric  $\gamma_k = \hat{\Psi}^* g$  where  $g$  is the canonical euclidian product on  $\mathbb{R}^{m+1}$ . Now the map  $\hat{\Gamma} : (\mathbb{R}^{m+1})^{k+1} \times (\mathbb{R}^{m+1})^{k+2}$  defined by

$$\Gamma(x_0, x_1, \dots, x_k, z) = (x_0, x_1, \dots, x_k, x_k + z)$$

is a diffeomorphism. So the restriction  $\Gamma$  of  $\hat{\Gamma}$  to  $\mathcal{C}^k(m) \times \mathbb{S}^m$  is a diffeomorphism  $\Gamma : \mathcal{C}^k(m) \times \mathbb{S}^m \rightarrow \mathcal{C}^{k+1}(m)$ . Finally,  $\bar{\Psi} = \Gamma \circ \hat{\Psi}$  induces, by restriction, a diffeomorphism  $\Psi : S(\mathcal{D}_k, \mathcal{C}^k(m), \gamma_k) \rightarrow \mathcal{C}^{k+1}(m)$  which commutes with the canonical the projections:

$$S(\mathcal{D}_k, \mathcal{C}^k(m), \gamma_k) \rightarrow \mathcal{C}^k(m) \text{ and } \mathcal{C}^{k+1}(m) \rightarrow \mathcal{C}^k(m).$$

On  $\mathcal{D}_k$ , we have a riemannian metric so that the global basis given in (20) is orthonormal. It follows that the map  $\Psi : S(\mathcal{D}_k, \mathcal{C}^k(m), \gamma_k) \rightarrow \mathcal{C}^{k+1}(m)$  is given by

$$\Psi((x_0, x_1, \dots, x_k, \nu)) = ((x_0, x_1, \dots, x_k, x_k + \nu)).$$

So, in the global chart of  $S(\mathcal{D}_k, \mathcal{C}^k(m), \gamma_k)$  defined by  $\Psi$ , according to (20), the spherical prolongation  $(\mathcal{D}_k)^{[1]}$  of  $\mathcal{D}_k$  is generated by the tangent space to the sphere centered at  $(x_0, \dots, x_k)$  and the vector field:

$$Y_{k+1} = \sum_{r=1}^{m+1} (x_{k+1}^r - x_k^r) (x_k^r - x_{k-1}^r) X_k + \sum_{r=1}^{m+1} (x_{k+1}^r - x_k^r) \Pi_k\left(\frac{\partial}{\partial x_k^r}\right)$$

According to (19) and the value of  $X_k$ , this vector field can be written

$$\begin{aligned} & \sum_{r=1}^{m+1} \{ (x_{k+1}^r - x_k^r)(x_k^r - x_{k-1}^r) [\sum_{i=0}^{k-1} \prod_{j=i+1}^k \mathcal{A}_j \mathcal{Z}_i + \mathcal{V}_k] + (x_{k+1}^r - x_k^r) [\frac{\partial}{\partial x_k^r} - (x_k^r - x_{k-1}^r) \mathcal{V}_k] \} \\ = & \sum_{r=1}^{m+1} \{ (x_{k+1}^r - x_k^r)(x_k^r - x_{k-1}^r) [\sum_{i=0}^{k-1} \prod_{j=i+1}^k \mathcal{A}_j \mathcal{Z}_i] + (x_{k+1}^r - x_k^r) \frac{\partial}{\partial x_k^r} \} \end{aligned}$$

Note that  $\sum_{r=1}^{m+1} (x_{k+1}^r - x_k^r) \frac{\partial}{\partial x_k^r} = \mathcal{Z}_k$ . So according to Remark 3.1, applied at level  $k+1$ , we have

$$Y_{k+1} = [\sum_{r=1}^{m+1} (x_{k+1}^r - x_k^r)(x_k^r - x_{k-1}^r) Y_k + Z_k.$$

and the distribution  $\mathcal{D}_{k+1}$  is generated by

$$\{ (x_{k+1}^r - x_k^r) Y_{k+1} + \frac{\partial}{\partial x_{k+1}^r} \mid r = 1 \cdots m+1 \}$$

Again from Lemma 3.2 part (2), at level  $k+1$  we get:

$$\Psi_*[(\mathcal{D}_k)^{[1]}] = \mathcal{D}_{k+1}$$

□

### Proof of Theorem 3.2

Note that on the tangent bundle  $T\mathbb{R}^{m+1}$ , we can put the global chart defined by the map  $(x_1, x_2) \rightarrow (x_1, x_2 - x_1)$ . On the other hand, the riemannian metric  $g_1$  on  $T\mathbb{R}^{m+1}$  induces by the canonical metric  $g$  is again the canonical metric on the product  $\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$ . It follows that  $S(T\mathbb{R}^{m+1}, \mathbb{R}^{m+1}) \subset T\mathbb{R}^{m+1}$  can be identified with  $\mathbb{R}^{m+1} \times S^k$  as submanifold of  $\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$ . So we have

$$T_{(x,\nu)} S(T\mathbb{R}^{m+1}, \mathbb{R}^{m+1}) = \{ (u, v) \in T_{(x,\nu)}(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}) \mid \text{such that } g(\nu, u) = 0 \}.$$

Recall that  $\mathcal{Z}_0 = \sum_{r=1}^{m+1} (x_1^r - x_0^r) \frac{\partial}{\partial x_0^r}$  on  $\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$ . In the previous coordinates, any tangent vector of  $\mathbb{R}^{m+1}$  at a point  $x_1$  can be written  $(x_1, x_2 - x_1)$ . So,  $\mathcal{Z}_0$  defines a global section of the unit bundle associated to  $T\mathbb{R}^{m+1}$

According to Lemma 3.2, the distribution  $\mathcal{D}_1$  is generated by

$\mathbb{R}\mathcal{Z}_0$  and  $T\mathbb{S}^m$  in  $T\mathbb{R}^{m+1} \times T\mathbb{S}^m \equiv TS(T\mathbb{R}^{m+1}, \mathbb{R}^{m+1})$ . If  $\Pi_1 : \mathbb{R}^{m+1} \times \mathbb{S}^m \rightarrow \mathbb{R}^{m+1}$  denote the natural projection, for any  $v = \lambda \mathcal{Z}_0 + w \in \{\mathcal{D}_1\}_{(x,\nu)}$  with  $w \in T\mathbb{S}^k$ , we have  $d\Pi_1(v) = \lambda \nu$  so,  $\Delta_1 = (T\mathbb{R}^{m+1})^{[1]}$  and we get the result for  $k=1$ .

Assume that we have a diffeomorphism  $F^k : \hat{P}^k(m) \rightarrow \mathcal{C}^k(m)$  which satisfies the properties (i), and (ii) of Theorem 2.4.

From Proposition 3.1, we have diffeomorphism  $\Psi : S(\mathcal{D}_k, \mathcal{C}^k(m), \gamma_k) \rightarrow \mathcal{C}^{k+1}(m)$  so that  $\Psi_*[(\mathcal{D}_k)^{[1]}] = \mathcal{D}_{k+1}$  and which commutes with the natural projections

$$S(\mathcal{D}_k, \mathcal{C}^k(m), \gamma_k) \rightarrow \mathcal{C}^k(m) \quad \text{and} \quad \mathcal{C}^{k+1}(m) \rightarrow \mathcal{C}^k(m)$$

According to previous induction, we can put on  $\hat{P}^k(m)$ , the riemannian metric  $\bar{\gamma}_k = (\Psi^k)^*(\gamma_k)$ . From Lemma 2.3, we can extend  $F^k : \hat{P}^k(m) \rightarrow \mathcal{C}^k(m)$  into a diffeomorphism  $\hat{\Phi}^k : S(\hat{\Delta}_k, \hat{P}^k(m), \bar{\gamma}_k) \rightarrow S(\mathcal{D}_k, \mathcal{C}^k(m), \gamma_k)$  such that  $\hat{\Phi}_*[(\hat{\Delta}_k)^{[1]}] = (\mathcal{D}_k)^{[1]}$  and which commutes with the natural projections

$$S(\hat{\Delta}_k, \hat{P}^k(m), \bar{\gamma}_k) \rightarrow \hat{P}^k(m) \quad \text{and} \quad \mathcal{C}^{k+1}(m) \rightarrow \mathcal{C}^k(m)$$

Finally, according to Lemma 2.2, when we put on  $\hat{P}^k(m)$  the riemannian metric induces by induction on the tower bundle (11), we also have a diffeomorphism

$\Phi : \hat{P}^{k+1}(m) \rightarrow S(\hat{\Delta}_k, \hat{P}^k(m), \bar{\gamma}_k)$  which commutes with the canonical projections

$$\hat{P}^{k+1}(m) \rightarrow \hat{P}^k(m) \quad \text{and} \quad S(\hat{\Delta}_k, \hat{P}^k(m), \bar{\gamma}_k) \rightarrow \hat{P}^k(m)$$

and such that  $\Phi_*(\hat{\Delta}_{k+1}) = \hat{\Delta}_k^{[1]}$ .

□

**Remark 3.2.** According to Theorem 2.4 and Theorem 3.2, from towers (12) and (17) we get the following diagram each vertical map is a 2-fold covering for  $k \geq 1$ :

$$\begin{array}{ccccccc} \cdots & \rightarrow & \mathcal{C}^k(m) & \rightarrow & \mathcal{C}^{k-1}(m) & \rightarrow & \cdots & \rightarrow & \mathcal{C}^1(m) & \rightarrow & \mathcal{C}^0(m) := \mathbb{R}^{m+1} \\ \cdots & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\ \cdots & \rightarrow & P^k(m) & \rightarrow & P^{k-1}(m) & \rightarrow & \cdots & \rightarrow & P^1(m) & \rightarrow & P^0(m) := \mathbb{R}^{m+1} \end{array}$$

### 3.3. Hyperspherical coordinates.

Consider the natural global diffeomorphism  $\mathcal{F}^k : \mathcal{C}^k(m) \rightarrow \mathbb{R}^{m+1} \times (\mathbb{S}^m)^k$  given by

$$\mathcal{F}^k(x_0, x_1, \dots, x_i, \dots, x_k) = (x_0, x_1 - x_0, \dots, x_i - x_{i-1}, \dots, x_k - x_{k-1})$$

Note that, according to Theorem 3.2, the map  $\mathcal{F}^k \circ F^k$  is also a global diffeomorphism from  $\hat{\mathcal{P}}^k(m)$  to  $\mathcal{S}^k(m) = \mathbb{R}^{m+1} \times (\mathbb{S}^m)^k$  and, if  $\varrho^k : \mathcal{S}^k \rightarrow \mathcal{S}^{k-1}$  is the canonical projection, we have the following commutative diagram:

$$(21) \quad \begin{array}{ccccc} & F^k & & \mathcal{F}^k & \\ \hat{\mathcal{P}}^k(m) & \rightarrow & \mathcal{C}^k(m) & \rightarrow & \mathcal{S}^k(m) \\ \downarrow \hat{\pi}^k & & \downarrow \rho^k & & \downarrow \varrho^k \\ \hat{\mathcal{P}}^{k-1}(m) & \xrightarrow{F^{k-1}} & \mathcal{C}^{k-1}(m) & \xrightarrow{\mathcal{F}^{k-1}} & \mathcal{S}^{k-1}(m) \end{array}$$

Via this global chart, each point  $q = (x_0, x_1, \dots, x_i, \dots, x_k) \in \mathcal{C}^k(m)$  could be identified with

$$\zeta = \mathcal{F}^k(q) = (x_0, z_1, \dots, z_i, \dots, z_k) \in \mathbb{R}^{m+1} \times (\mathbb{S}^m)^k.$$

We will put on each factor  $\mathbb{S}^m$  charts given by *hyperspherical coordinates*. We first recall some basic facts about this type of coordinates.

The *hyperspherical coordinates* in  $\mathbb{R}^{m+1}$  are given by the relations:

$$(22) \quad \begin{cases} z^1 = \rho \phi^1(\theta) = \rho \sin \theta^1 \cdots \sin \theta^{m-1} \sin \theta^m \\ z^2 = \rho \phi^2(\theta) = \rho \sin \theta^1 \cdots \sin \theta^{m-1} \cos \theta^m \\ z^3 = \rho \phi^3(\theta) = \rho \sin \theta^1 \cdots \sin \theta^{m-2} \cos \theta^{m-1} \\ \dots \\ z^k = \rho \phi^k(\theta) = \rho \sin \theta^1 \cos \theta^2 \\ z^{k+1} = \rho \phi^{k+1}(\theta) = \rho \cos \theta^1 \end{cases}$$

with  $\rho^2 = (z^1)^2 + \dots + (z^{k+1})^2$ ,  $0 \leq \theta^k \leq 2\pi$  and  $0 \leq \theta^j \leq \pi$  for  $1 \leq j \leq m-1$ .

We consider  $\hat{\Phi}(\rho, \theta) = \rho \Phi(\theta) = z$ , the application from  $]0, +\infty[ \times [0, \pi] \times \dots \times [0, \pi] \times [0, 2\pi]$  to  $\mathbb{R}^{m+1}$ .

The jacobian matrix  $D\hat{\Phi}$  of  $\hat{\Phi}$  is:

$$D\hat{\Phi} = \left( \phi \quad \rho \frac{\partial \phi}{\partial \theta^1} \quad \dots \quad \rho \frac{\partial \phi}{\partial \theta^m} \right),$$

where  $\phi$  (resp.  $\frac{\partial \phi}{\partial \theta^j}$ ) is the column vector of components  $\{\phi^1, \dots, \phi^{m+1}\}$  (resp.  $\{\frac{\partial \phi^1}{\partial \theta^j}, \dots, \frac{\partial \phi^{k+1}}{\partial \theta^j}\}$ ).

The inverse of this matrix is then the transpose of the matrix:

$$(23) \quad \left( \phi \quad \frac{1}{\rho \|\frac{\partial \phi}{\partial \theta^1}\|} \frac{\partial \phi}{\partial \theta^1} \quad \dots \quad \frac{1}{\rho \|\frac{\partial \phi}{\partial \theta^m}\|} \frac{\partial \phi}{\partial \theta^m} \right)$$

We come back to  $\mathcal{S}^k = \mathbb{R}^{m+1} \times (\mathbb{S}^m)^k$  which is considered as a subset in  $(\mathbb{R}^{m+1})^{k+1}$ . For  $i = 1, \dots, k$ , let  $\mathbb{S}_i$  be the canonical sphere in the  $i^e$  factor  $\mathbb{R}_i^{m+1}$ . Given a point  $\alpha$  in the sphere  $\mathbb{S}_i$ , there exists hyperspherical coordinates  $z_i = \hat{\Phi}_i(\rho_i, \theta_i) = \rho_i \Phi_i(\theta_i^1, \dots, \theta_i^k)$  defined for  $0 \leq \theta_i^k \leq 2\pi$  and  $0 < \theta_i^j < \pi$ ,  $j = 1, \dots, m-1$ , where  $\Phi_i(0, \dots, 0) = \alpha$ . So, for a given point  $\zeta = (x_0, z_1, \dots, z_i, \dots, z_k) \in \mathcal{S}^k$ , we get a chart  $\mathcal{H}^k = (Id - x_0, (\hat{\Phi}_1)^{-1}, \dots, (\hat{\Phi}_2)^{-1}, \dots, (\hat{\Phi}_k)^{-1})$  centered at  $\zeta$ , such that its restriction to  $\rho_i = 1$ ,  $i = 1, \dots, k$ , induces a chart of  $\mathcal{S}^k$  (centered at  $\zeta$ ).

We can note the map  $\mathcal{H}^k = (Id - x_0, (\hat{\Phi}_1)^{-1}, (\hat{\Phi}_2)^{-1}, \dots, (\hat{\Phi}_k)^{-1})$  is a hyperspherical chart on  $\mathcal{S}^k(m)$  then we have

#### Definition 3.1.

- (1) For any  $\zeta \in \mathcal{S}^k(m)$  any map of type  $\mathcal{H}^k$  around  $\zeta$  is called a *hyperpspherical chart* on  $\mathcal{S}^k(m)$
- (2) For any  $q = (\mathcal{F}^k)^{-1}(\zeta)$  in  $\mathcal{C}^k(m)$ , any map of type  $\mathcal{H}^k \circ \mathcal{F}^k$  around  $q$  is called a *hyperspherical chart* on  $\mathcal{C}^k(m)$ .
- (3) for any  $p = (\mathcal{F}^k \circ F^k)^{-1}(\zeta)$  on  $\hat{\mathcal{P}}^k(m)$ , any map of type  $\mathcal{H}^k \circ \mathcal{F}^k \circ F^k$  around  $p$  is called a *hyperspherical chart* on  $\hat{\mathcal{P}}^k(m)$ .

Now, we introduce the following notations in hyperspherical coordinates:

#### Notations 3.1.

- $A_i = \sum_{r=1}^{m+1} \phi_{i-1}^r \phi_i^r$  for  $i = 1, \dots, k-1$  and  $A_k = 1$ ;

$$\begin{aligned}
& \bullet Z_0 = \sum_{r=1}^{m+1} \phi_0^r \frac{\partial}{\partial x_0^r} \\
& \bullet Z_i = \sum_{j=1}^m B_i^j \frac{\partial}{\partial \theta_{i-1}^j} \text{ for } i = 1, \dots, k-1 \\
& \text{with:} \\
& \bullet B_i^1 = \sum_{r=1}^{m+1} \frac{\partial \phi_{i-1}^r}{\partial \theta_{i-1}^1} \phi_i^r \quad \text{for } i = 1, \dots, k-1 \\
& \bullet B_i^j = \frac{1}{\|\frac{\partial \phi_{i-1}^r}{\partial \theta_{i-1}^j}\|} \sum_{r=1}^{m+1} \frac{\partial \phi_{i-1}^r}{\partial \theta_{i-1}^j} \phi_i^r \quad \text{for } i = 1, \dots, k-1 \text{ and } j = 2, \dots, m \\
& \bullet X_l^i = \frac{\partial}{\partial \theta_l^i}, \text{ for } i = 1, \dots, m \text{ and } 0 \leq l \leq k-1 ; \\
& \bullet X_l^0 = \sum_{i=0}^l f_l^i Z_i \text{ for } 0 \leq l \leq k-1 ; \\
& \text{with } f_l^i = \prod_{j=i+1}^l A_j, \text{ for } i = 0, \dots, l-1, \quad 0 \leq l \leq k-1 \text{ and } f_l^l = 1.
\end{aligned}$$

**Remark 3.3.** In Lemma 3.1

we already defined a function  $\mathcal{A}_j(q) = - \langle \mathcal{N}_j(q), \mathcal{N}_{j-1}(q) \rangle$ . It is clear that that we have:

$$A_j \circ (\mathcal{H}^k \circ \mathcal{F}^k) = \mathcal{A}_j$$

**Theorem 3.4.** For any  $k \geq 1$  we have

- (1) In hyperspherical coordinates, on each manifold  $\mathcal{S}^k(m)$ ,  $\mathcal{C}^k(m)$  and  $\hat{\mathcal{P}}^k(m)$ , the corresponding distributions  $F_*^k(\mathcal{D}_k)$ ,  $\mathcal{D}_k$  and  $\hat{\Delta}_k$  respectively, is generated by  $\{X_{k-1}^0, X_{k-1}^1, \dots, X_{k-1}^m\}$ .
- (2) we have a tower of commutative diagrams:

$$\begin{array}{ccccccccc}
\cdots & \rightarrow & \hat{\mathcal{P}}^k(m) & \rightarrow & \hat{\mathcal{P}}^{k-1}(m) & \rightarrow & \cdots & \rightarrow & \hat{\mathcal{P}}^1(m) & \rightarrow & \hat{\mathcal{P}}^0(m) := \mathbb{R}^{m+1} \\
& & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
\cdots & \rightarrow & \mathcal{C}^k(m) & \rightarrow & \mathcal{C}^{k-1}(m) & \rightarrow & \cdots & \rightarrow & \mathcal{C}^1(m) & \rightarrow & \mathcal{C}^0(m) := \mathbb{R}^{m+1} \\
& & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
(24) \quad \cdots & \rightarrow & \mathcal{S}^k(m) & \rightarrow & \mathcal{S}^{k-1}(m) & \rightarrow & \cdots & \rightarrow & \mathcal{S}^1(m) & \rightarrow & \mathcal{S}^0(m) := \mathbb{R}^{m+1} \\
& & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
\cdots & \rightarrow & P^k(m) & \rightarrow & P^{k-1}(m) & \rightarrow & \cdots & \rightarrow & P^1(m) & \rightarrow & P^0(m) := \mathbb{R}^{m+1}
\end{array}$$

where, for all  $l = 1, \dots, k$ ,

- at each horizontal level the horizontal map between column number  $l$  to column number  $l-1$  is a fibration in sphere (resp. projective space) for the three first line (resp. for the last line)
- Each vertical map sends each corresponding distribution of index  $l$  onto the corresponding distribution  $l$  on the lower line. Moreover the three first vertical maps are diffeomorphisms and the last vertical ones are two-fold coverings.

*Proof.*

In [15] section 5, it is proved that, in hyperspherical coordinates, the distribution  $F_*^k(\mathcal{D}_k)$  is precisely generated by  $\{X_{k-1}^0, X_{k-1}^1, \dots, X_{k-1}^m\}$ . According to Theorem 3.2, the diffeomorphism  $F^k : \hat{\mathcal{P}}^k(m) \rightarrow \mathcal{C}^k(m)$  sends the distribution  $\hat{\Delta}_k$  onto  $\mathcal{D}_k$ . This ends the proof of part (1). The part (2) is a consequence of (12), Theorem 3.2, diagram (21) and part (1).  $\square$

**Remark 3.4.**

All manifolds which appear in the towers (24) are analytic manifolds and all maps in these towers are also analytic.

#### 4. RC - RVT CODES AND CONFIGURATION OF ARTICULATED ARMS

##### 4.1. RC and RVT codes according to [4].

In this subsection we expose the theory of **RC** and **RVT** codes introduced in [4] and we also adapt it to the context of spherical prolongation.

Let be  $D$  a distribution of constant rank on a manifold  $M$  fitted with a Riemannian metric. We will denote indistinctly by  $\mathcal{P}(M, D)$  the sphere bundle  $S(M, D, g)$  or the projective bundle  $P(M, D)$  and by  $D^{\{1\}}$  the spherical prolongation or the Cartan prolongation of  $D$  on  $\mathcal{P}(M, D)$ . By induction the corresponding tower of bundle ( cf (7) and (11))

$$(25) \quad \cdots \rightarrow \mathcal{P}^k(M) \rightarrow \mathcal{P}^{k-1}(M) \rightarrow \cdots \rightarrow \mathcal{P}^1(M) \rightarrow \mathcal{P}^0(M) := M$$

fitted at each level  $j$  with a distribution denoted by  $\mathfrak{D}_k = (\mathfrak{D}_{k-1})^{\{1\}}$ .

When  $M = \mathbb{R}^{m+1}$  we simply denote  $\mathcal{P}^k(m)$  any one space  $\hat{P}^k(m)$  or  $P^k(m)$ .

For any  $k \geq 1$  we denote by  $\Pi^k$  the natural projection of  $\mathcal{P}^k(m)$  onto  $\mathcal{P}^{k-1}(m)$ . The tangent bundle to the fiber of  $\Pi^k$  is the **vertical bundle** denoted  $V_k$  and by construction we have  $V_k(p) \subset \mathfrak{D}_k(p)$ .

For any  $p \in \mathcal{P}^{k-1}(m)$ , the fiber  $(\Pi^k)^{-1}(p)$  is denoted  $S^k(p)$ . So for such a point  $p$ , from (25), we get a tower of fiber bundles

$$(26) \quad \cdots \rightarrow \mathcal{P}^l(S^k(p)) \rightarrow \mathcal{P}^{l-1}(S^k(p)) \rightarrow \cdots \rightarrow \mathcal{P}^1(S^k(p)) \rightarrow \mathcal{P}^0(S^k(p)) = S^k(p)$$

Coming back to our general context, on each  $\mathcal{P}^j(S^k(p))$ , we have a distribution  $\mathfrak{d}_j^k$  again inductively defined by  $\mathfrak{d}_j^k = [\delta_{j-1}^k]^{\{1\}}$ . Such a tower will be called a **fiber prolongation tower**. Note that, in such a tower, the differential system  $(\mathcal{P}^j(S^k(p)), \mathfrak{d}_j^k, q)$  is always equivalent to  $(\mathcal{P}^l(m-1), \mathfrak{D}_l, p)$  for some adequate point  $p \in \mathcal{P}^l(m-1)$ .

Of course we have  $\mathcal{P}^l(S^k(p)) \subset \mathcal{P}^{k+l}(m)$  and  $\mathfrak{d}_j^k(q)$  is an hyperplane in  $\mathfrak{D}_{k+j}(q)$  for any  $q \in \mathcal{P}^j(S^k(p))$ . In particular  $V_k(q)$  is nothing but that  $\mathfrak{d}_0^k(q)$  for any  $q \in S^k(p)$

On the other hand, for any  $k > l \geq 0$ , let be  $\Pi^{kl}$  the natural projection of  $\mathcal{P}^k(m)$  onto  $\mathcal{P}^l(m)$  which is the composition  $\Pi^k \circ \Pi^{k-1} \circ \cdots \circ \Pi^{l+1}$ . If  $p$  is a point in  $\mathcal{P}^k(M)$ , we denote by  $p_l$  its projection  $p_l = \Pi^{kl}(p) \in \mathcal{P}^l(m)$  and we say that  $p_l$  is **under**  $p_k = p$ . With these notations, for  $k \geq 1$ , each point  $p_k \in \mathcal{P}^k(m)$  can be written  $(p_{k-1}, z)$  for some  $z \in S^k(p_{k-1})$ .

So, at each level  $k \geq 1$ , inside the space  $\mathfrak{D}_k(p)$ , we have the family of hyperplanes  $\mathfrak{d}_j^i(p)$ , with  $i + j = k$ , coming from a fiber prolongation of order  $j$  of the tangent space of the fiber  $S^i(p_{i-1})$  for  $i = 1, \dots, k$ .

Recall that a family  $\{E_i\}_{i \in I}$  of hyperplanes of  $\mathbb{R}^N$  is in general position if, for every subset  $J \subset I$  of indices whose cardinality  $|J| \leq N$  we have that the codimension of the intersection  $\bigcap_{i \in J} E_i$  is exactly  $|J|$ .

**Theorem 4.1.** [4]

*the family of hyperplanes  $\mathfrak{d}_j^i(p)$ , with  $i + j = k$  is in general position inside the space  $\mathfrak{D}_k(p)$*

*Proof.* This result is proved in [4] for Cartan prolongation (Theorem 6.1). If  $\hat{p}$  is a point in  $\hat{P}^k(m)$  we denote by  $p$  its projection  $\tau^k(\hat{p})$  in  $P^k(m)$  (cf Theorem 2.4). According to Theorem 2.4, each hyperplane  $\mathfrak{d}_j^i(\hat{p})$  in  $\mathfrak{D}_k(\hat{p})$  project via  $\tau^k$  onto any hyperplane  $\mathfrak{d}_j^i(p)$  in  $\Delta_k(p)$  corresponding to the previous process of fiber prolongation in the tower of bundles (11). By applying result of [4] we get the proof in the context of spherical prolongations.  $\square$

According to [4] and [5] we have the following Definitions

**Definition 4.1.**

- (1) any hyperplane  $\mathfrak{d}_j^i(p)$ , with  $i + j = k$  in the space  $\mathfrak{D}_k(p)$  is called a **critical hyperplane** at  $p$ . A direction  $l$  or a vector  $v$  in  $\mathfrak{D}_k(p)$  is called **critical** if it lies in one critical hyperplane. Otherwise  $l$  (resp.  $v$ ) is called **regular**. Moreover a critical direction  $l$  or a vector  $v$  in  $\mathfrak{D}_k(p)$  is called **vertical** (resp. **tangency**) if the singular hyperplane which contains it is  $V_k(p) = \mathfrak{d}_0^k(p)$  (resp.  $\mathfrak{d}_j^i(\hat{p})$  for  $i > 0$ ).
- (2) A point  $p = (p_{k-1}, z) \in \mathcal{P}^k(m)$  is called **regular, critical, vertical, tangency** if  $z \in \mathfrak{D}_{k-1}(p_{k-1})$  is respectively regular, critical, vertical, tangency.

**Remark 4.1.**

- (1) let be  $\hat{p} \in \hat{P}^k(m)$  and  $p = \tau^k(\hat{p}) \in P^k(m)$ . It follows from Theorem 2.4 that  $\hat{p}$  is regular, critical, vertical, tangency if and only if  $p$  is respectively regular, critical, vertical, tangency. In the converse, for any  $p \in P^k(m)$ , each points in  $\tau^k(p) \subset \hat{P}^k(m)$  have the same previous qualification as  $p$

- (2) Inside any fiber prolongations tower (26) we can consider a fiber prolongation tower from some fiber of the projection  $\mathcal{P}^l(S^k(p)) \rightarrow \mathcal{P}^{l-1}(S^k(p))$  and look for the corresponding critical hyperplane in  $\mathfrak{d}_i^k(q)$ . Then such a critical hyperplane is in fact an intersection of type  $\mathfrak{d}_i^k(q) \cap \mathfrak{d}_j^i(q)$  with  $i > k$  and  $k + l = i + j$  (see Proposition 6.2 of [4]).
- (3) if a point  $p = (p_{k-1}, z) \in \mathcal{P}^k(m)$  is critical, than  $z$  can belongs to the intersection of many critical hyperplanes and **not only one** critical hyperplane.

The **RC** code of a point  $p \in \mathcal{P}^k(m)$  is a word  $\sigma = \sigma_1 \cdots \sigma_l \cdots \sigma_k$  whose letter  $\sigma_l$  is  $R$  (resp.  $C$ ) if the point  $p_l$  under  $p$  is regular (resp. critical). Note that, by convention, the first letter is always  $R$ .

The **RVT** code of a point  $p \in \mathcal{P}^k(m)$  is a word  $\omega = \omega_1 \cdots \omega_l \cdots \omega_k$  obtained from its **RC**  $\sigma(p)$  such that:

- $\omega_i = R$  if  $\sigma_i = R$ ;
- $\omega_i = V$  if  $\sigma_i = C$  and the point  $p_i$  under  $p$  is vertical;
- $\omega_i = T$  if  $\sigma_i = C$  and the point  $p_i$  under  $p$  is tangency.

**Remark 4.2.**

- (1) According to Remark 4.1, the **RC** (resp. **RVT**) code of any point  $\hat{p} \in \hat{P}^k(m)$  is the same as the **RC** (resp. **RVT**) code of its projection  $p = \tau^k(\hat{p}) \in P^k(m)$ .
- (2) The **RC** code gives rise to a partition of  $\mathcal{P}^k(m)$  into  $2^{k-1}$  set of points which have the same **RC** code  $\sigma$ . Let be  $\hat{C}_\sigma$  (resp.  $C_\sigma$ ) the set of point  $\hat{p} \in \hat{P}^k(m)$  (resp.  $p \in P^k(m)$ ) whose **RC** code is  $\sigma$ . Then  $\tau^k(\hat{C}_\sigma) = C_\sigma$  and  $(\tau^k)^{-1}(C_\sigma) = \hat{C}_\sigma$ .
- (3) Remark 4.1 part (2) imposes that each  $p_i$  which is tangency,  $p_i$  must lies in a fiber tower prolongations for some  $p_j$  under  $p_i$ . So the first letter  $C$  which appears in the **RC** codes, says level  $i$ , imposes that  $p_i$  must vertical.
- (4) Each **RC** code  $\sigma$  generates, a priori,  $2^{n_\sigma}$  **RVT** codes  $\omega$ , if  $n_\sigma$  is the number of letters  $C$  in  $\sigma$ . However, from part (3), a letter  $T$  cannot follows immediately a letter  $R$  in such a code because each tangency point must lies in a prolongation tower of some point  $p_j$  under  $p_i$ . So, after a letter  $R$  the first eventually letter  $T$  which appear, we must at most one letter  $V$  between these letters.
- (5) According to Remark 4.1 (3), for a critical point  $p = (p_{k-1}, z) \in \mathcal{P}^k(m)$ , when  $z$  can belongs to the intersection of many critical hyperplanes the **RVT** code generated by the **RC** code can be not well defined. In this case, when  $z$  belongs is not vertical and belongs to one (and only one) of them, we need much precision in the code about the possible letters "T" for instance  $T_1, T_2 \cdots, T_\nu$ . Moreover, much more complicated codification is needed if  $z$  belongs to a the intersection of many critical hyperplanes. For instance, if this intersection is a line we can use a codification by letters  $L_1, L_2 \cdots$  as it is proposed in [4] and [5].

In the **RVT** code, according to Remark 4.2 (5), and [4] and [5]

- if  $z$  belongs to only one critical hyperplane we will use letters  $V, T_1, T_2 \cdots$ , in the **RVT** code;
- if  $z$  belongs to the intersection of exactly two critical hyperplanes referenced  $T_i$  and  $T_j$  we will use letters of type  $T_{ij}$  in the **RVT** code. Moreover **we adopt the following convention** :  
 $T_0$  is always relative to a **vertical** hyperplanes and  $T_i$ , for  $i > 0$  to a **critical** hyperplanes which is **not vertical**.

- more generally, if  $z$  belongs to the intersection of exactly  $n$  critical hyperplanes referenced  $T_0 = V$  and  $T_{i_1}, \cdots, T_{i_{n-1}}$  or  $T_{i_1}, \cdots, T_{i_n}$  with  $i_1 \cdots i_n \neq 0$  we will use letters of type  $T_{0i_1 \cdots i_n}$  or  $T_{i_1 \cdots i_n}$  in the **RVT** code.

Note that, in such a notation, in a **RVT** code, a letter  $T_0$  always means "vertical" and each letters  $T_i$  with  $i > 0$  means "tangency".

**Definition 4.2.**

we will say that a word  $\omega$  (resp. a class  $C_\omega$ ) in **RVT** code is of depth  $d$  if this word  $\omega$  contains at least one letter of type  $T_{i_1 \cdots i_d}$  with  $i_1 \geq 0$ .

For instance, for  $m = 3$  and  $1 \leq k \leq 4$  the different **RVT** classes can be encoding in the following way (compare with [4] before Corollary 4.48):

- $k=1$ :  $R$
- $k=2$ :  $RR, RV$
- $k=3$ :  $RRR, RRV, RVV, RVR, RVT, RT_0T_{01}$
- $k=4$ :

$$\begin{aligned}
& RRRR, RRRV, \\
& RRVR, RRVV, RRVV, RRT_0T_{01} \\
& RVRR, RVRV, RVVR, RVVV, RVVT, RVT_0T_{01} \\
& RVTR, RVTV, RVTT, RVRT_{01}, RVTT_{01} \\
& RT_0T_{01}R, RT_0T_{01}V, RT_0T_{01}T_1, RT_0T_{01}T_2, RT_0T_{01}T_{01}, RT_0T_{01}T_{02}, RT_0T_{01}T_{12}
\end{aligned}$$

All words  $RT_0T_{01}$ ,  $RRT_0T_{01}$ ,  $RVT_0T_{01}$ ,  $RVRT_{01}$ ,  $RTT_{01}R$ ,  $RT_0T_{01}V$ ,  $RT_0T_{01}T_1$ ,  $RT_0T_{01}T_2$ ,  $RT_0T_{01}T_{01}$ ,  $RT_0T_{01}T_{02}$ ,  $RT_0T_{01}T_{12}$  are of depth 2. The other words are of depth 1

#### 4.2. Vertical points and configurations of articulated arms.

Let be  $q = (x_0, \dots, x_k) \in \mathcal{C}^k(m)$  and  $\hat{p} = (F^k)^{-1}(q)$ . For  $0 \leq l \leq k$ , we denote by  $q_l = F^l(\hat{p}_l)$  for any point  $\hat{p}_l$  under  $\hat{p}$ . In fact, according to Theorem 3.2, we have  $q_l = (x_0, \dots, x_l) = \rho^{k,l-1} = \rho^k \circ \dots \circ \rho^{l-1}(q)$ . We also say that  $\{q_l, l = 0, \dots, k-1\}$  are **points under**  $q$ . Moreover we can write  $q = (q_{k-1}, x_k)$  for  $x_k \in (\rho^k)^{-1}(q_{k-1})$  and, again from Theorem 3.2,  $\hat{p} = (F^k)^{-1}(q)$  is vertical if and only if the direction generated by  $x_k - x_{k-1}$  is vertical according to the projection  $\rho^k : \mathcal{C}^k(m) \rightarrow \mathcal{C}^{k-1}(m)$ .

More generally we can transpose the characterization of points of  $\hat{P}^k(m)$  onto points of  $\mathcal{C}^k(m)$ . More precisely we have:

**Definition 4.3.** Consider a point  $q = (F^k)^{-1}(\hat{p}) \in \mathcal{C}^k(m)$

- (1)  $q$  is called **regular, critical, vertical, tangency** if  $\hat{p}$  is regular, critical, vertical, tangency respectively
- (2) the code of  $q$  will be the code of the corresponding point  $\hat{p}$

At first, we have the following characterization of vertical points in  $\mathcal{C}^k(m)$

**Proposition 4.1.** Let be  $q \in \mathcal{C}^k(m)$ .

- (1) For all  $2 \leq l \leq k-1$ , We have the following equivalent properties:
  - (a) let be
$$\begin{aligned}
& [\mathcal{D}_k]_l \quad \subset \quad [\mathcal{D}_k]_{l-1} \\
& \quad \cup \quad \quad \cup \\
& L([\mathcal{D}_k]_{l-1}) \quad \subset \quad L([\mathcal{D}_k]_{l-2})
\end{aligned}$$
the sandwich number  $l$  associated to  $\mathcal{D}_k$ . Then  $[\mathcal{D}_k]_l(q) \subset L([\mathcal{D}_k]_{l-2})(q)$ .
  - (b)  $\mathcal{A}_{l-1}(q) = 0$
  - (c) the configuration  $q$  of the articulated arm  $(M_0, \dots, M_k)$  is such that the segments  $[M_{l-2}, M_{l-1}]$  and  $[M_{l-1}, M_l]$  are orthogonal in  $M_{l-1}$ .
  - (d)  $q_l$  is vertical
- (2)  $q$  is a Cartan point if and only if each point  $q_l$  under  $q$  is regular.

**Remark 4.3.** : according to our definition of a **singular points** (see the end of section 2.1), the Proposition 4.1 implies that a point  $q \in \mathcal{C}^k(m)$  is singular if and only if there exists a point  $q_l$  under  $q$  which is tangency

A consequence of this Proposition is the following:

#### Theorem 4.2.

- (1) The set  $\mathcal{C}_S$  of **singular points** of  $\mathcal{C}^k(m)$  is a subanalytic set of codimension 1. In particular, the set of Cartan points  $\mathcal{C}_C^k(m) = \mathcal{C}^k(m) \setminus \mathcal{C}_S$  is an open dense set.
- (2) Let be  $\omega$  a word of length  $k$  in letters  $R$  and  $V$  and denote by  $\{l_1, \dots, l_r\}$  the set of index  $\{l \in \{1, \dots, k\}\}$  such that  $\omega_l = V$ . We have the following properties:
  - (i) The set  $\mathcal{C}(\omega)$  of points  $q \in \mathcal{C}^k(m)$  whose **RVT** code is  $\omega$  is an analytic submanifold of  $\mathcal{C}^k(m)$  whose codimension is  $r$ .
  - (ii) the configuration of an articulated arm  $(M_0, \dots, M_k)$  belongs to  $\mathcal{C}(\omega)$  if and only if the unique consecutive segments  $[M_{l-2}, M_{l-1}]$  and  $[M_{l-1}, M_l]$  which are orthogonal in  $M_{l-1}$  are exactly for  $l = l_1, \dots, l_r$ .

#### Remark 4.4.

- (1) Our original definition of Cartan points (see the end of section 2.1) is somewhat different from [4] or [5]. However part (2) of Proposition 4.1 proves the equivalence of these definitions.
- (2) The result of part (1) of Theorem 4.2 is well known (see [4], [5], [6], [7], [14]).
- (3) Part (2) is also proved in [14] with an another notation for this set.

The proof of the Proposition 4.1 needs the following Lemma:

#### Lemma 4.1.



(1) For  $2 \leq l \leq k$  let be

$$\begin{array}{ccc} [\mathcal{D}_k]_l & \subset & [\mathcal{D}_k]_{l-1} \\ \cup & & \cup \\ L([\mathcal{D}_k]_{l-1}) & \subset & L([\mathcal{D}_k]_{l-2}) \end{array}$$

the sandwich number  $l$  associated to  $\mathcal{D}_k$ .

Then in hyperpshpherical coordinates,  $A_{l-1}(q) = 0$  if and only if  $[\mathcal{D}_k]_l(q) \subset L([\mathcal{D}_k]_{l-2})(q)$ .

(2) For  $2 \leq l \leq k$ ,  $q_l$  vertical if and only if  $A_{l-1}(q) = 0$

*Proof.* We place ourselves in the context of Notations 3.1.

By a simple calculation in hyperpshpherical coordinates, we obtain that the member  $[\mathcal{D}_k]_l$  of the multi-flag associate to  $\mathcal{D}_k$  is generated by

$$\{X_{l-1}^0, X_{l-1}^1, \dots, X_{l-1}^m\} \cup \{X_i^j, j = 1, \dots, m \mid l-2 \leq i \leq k-1\}$$

and  $L([\mathcal{D}_k]_{l-2})$  is generated by  $\{X_i^j, j = 1, \dots, m \mid l-2 \leq i \leq k-1\}$ .

On the other hand, we also have

$$(27) \quad X_{l-1}^0 = A_{l-1} X_{l-2}^0 + Z_{l-1}$$

As by construction, each  $Z_{l-1}$  belongs to  $L([\mathcal{D}_k]_{l-2})$ , so, if  $A_{l-1}(q) = 0$ , it follows that  $[\mathcal{D}_k]_l(q) \subset L([\mathcal{D}_k]_{l-2})(q)$ . On the other hand, as  $\{X_{l-2}^0, X_{l-2}^1, \dots, X_{l-2}^{m+1}\}$  is a basis of  $\mathcal{D}_{l-1}$  at  $q_{l-1}$  and  $Z_{l-1}$  is a linear combination  $\{X_{l-2}^1, \dots, X_{l-2}^{m+1}\}$  we must have  $X_{l-2}^0(q) \neq 0$ . So we get:  $[\mathcal{D}_k]_l(q) \subset L([\mathcal{D}_k]_{l-2})(q)$  if and only if  $A_{l-1}(q) = 0$ . According to Remark 3.3, this ends the proof of part (1).

According to Proposition 3.1, denote by  $\Psi^{l-1}$  the diffeomorphism from  $S(\mathcal{D}_{l-1}, \mathcal{C}^{l-1}(m), \gamma_{l-1})$  onto  $\mathcal{C}^l(m)$  constructed in this Proposition. We can write  $\Psi^{-1}(q_l) = (q_{l-1}, w_l)$  where  $w_l$  is a vector of norm 1 in  $\mathcal{D}_{l-1}(q_{l-1})$ . On the other hand,  $\mathcal{D}_{l-1}$  is generated by  $\{(x_{l-1}^r - x_{l-2}^r)X_{l-1} + \Pi_{l-1}(\frac{\partial}{\partial x_{l-1}^r}), r = 1, \dots, m+1\}$  (see Lemma 3.2). As these vector fields is an orthonormal family for the metric  $\gamma_{l-1}$ , we can write:

$$w_l = \sum_{r=1}^{m+1} z_l^r [(x_{l-1}^r - x_{l-2}^r)X_{l-1} + \Pi_{l-1}(\frac{\partial}{\partial x_{l-1}^r})]$$

Moreover, according to this decomposition and from the definition of  $\Psi^{l-1}$  in the proof of Proposition 3.1, we have

$$\Psi^{l-1}(q_{l-1}, w_l) = (x_0, \dots, x_{l-1}, x_{l-1} + z_l)$$

where  $q_{l-1} = (x_0, \dots, x_{l-1})$  and  $z_l = (z_l^1, \dots, z_l^{m+1})$ .

As  $\{\Pi_{l-1}(\frac{\partial}{\partial x_{l-1}^r}), r = 1, \dots, m+1\}$  generates the tangent space to each fiber of the projection

$$\mathcal{C}^{l-1}(m) \rightarrow \mathcal{C}^{l-2}(m), q_l \text{ is vertical if and only if } \sum_{r=1}^{m+1} (x_l^r - x_{l-1}^r)(x_{l-1}^r - x_{l-2}^r) = 0$$

But the first member of the previous relation is exactly  $\mathcal{A}_{l-1}(q)$ .

□

*Proof of Proposition 4.1*

The equivalence (a)  $\Leftrightarrow$  (b) comes from Lemma 4.1 part (1).

Also the equivalence (b)  $\Leftrightarrow$  (c) comes from part (2) of Lemma 4.1.

From Lemma 4.1 part (2), we get (d)  $\Leftrightarrow$  (b).

Now, a point  $q$  is a Cartan point if and only if  $[\mathcal{D}_k]_l(q)$  is not contained in  $L([\mathcal{D}_k]_{l-2})(q)$  for all  $2 \leq l \leq k$  (see the end of section 2.1). So, for part (1) if  $q$  is a Cartan point, no point  $q_l$  under  $q$  is vertical. If one  $p_l$  was tangency, from Remark 4.2 part (3), there must exists a point  $\hat{p}_j$  under  $\hat{p}_l$  which is vertical. So, again from part (1),  $q$  cannot be a Cartan point. We conclude that any point  $q_l$  under  $q$  is regular. The converse comes clearly from part (1).

□

*Proof of Theorem 4.2*

If  $q$  is singular, from part (2) of Proposition 4.1, there must exists a point  $q_l$  under  $q$  which is vertical. So

$$\text{the equation of set } \mathcal{C}_S \text{ is } \prod_{l=1}^{k-1} \mathcal{A}_l = 0.$$

Note that at, a point  $q$ , we have

$$\frac{\partial \mathcal{A}_l}{\partial x_{l+1}^r} = x_l^r - x_{l-1}^r \text{ for } r = 1 \dots, m+1$$

So, if  $\mathcal{A}_l(q) = 0$  we must have  $\frac{\partial \mathcal{A}_l}{\partial x_{l+1}^r}(q) \neq 0$  for some  $1 \leq r \leq m+1$ . According to Remark 3.4, it follows that  $\mathcal{C}_S$  is a subanalytic subset of  $\mathcal{C}^k(m)$  of codimension 1, which ends the proof of part (1).

Note that  $q$  belongs to  $\mathcal{C}(\omega)$  if and only if each points  $q_{l_1}, \dots, q_{l_r}$  under  $q$  are vertical. So, from property (b) of Proposition 4.1, the equations of  $\mathcal{C}(\omega)$  are:

$$(28) \quad \mathcal{A}_i(q) = 0 : \text{ for } i+1 = l_1, \dots, l_r$$

On the other hand each  $\mathcal{A}_l$  depends only of the variables  $x_{l-1}, x_l, x_{l+1}$ . So using the previous argument "  $\frac{\partial \mathcal{A}_l}{\partial x_{l+1}^r}(q) \neq 0$  for some  $1 \leq r \leq m+1$ ", the equations in (28) are independent. According again to Remark 3.4, it follows that  $\mathcal{C}(\omega)$  is an analytic submanifold of  $\mathcal{C}^k(m)$  of codimension  $r$ .

Part (2) of Theorem 4.2 is a direct consequence of property (c) of Proposition 4.1. □

#### 4.3. Tangency points and configurations of articulated arms.

We will prove the fundamental following results for tangency points  $q \in \mathcal{C}^k(m)$  :

##### Theorem 4.3.

- (1) *Let be  $q \in \mathcal{C}^k(m)$  a tangency point. Then, there exists  $2 \leq i \leq k-1$  such that the point  $q_i$  under  $q$  is vertical. Moreover, let be*

$$l = \sup\{2 \leq i \leq k-1 \text{ such that } q_i \text{ is vertical} \}$$

*Then if  $l < k$ , for any  $l < j \leq k$  the point  $q_j$  under  $q$  is tangency.*

- (2) *Denote by  $R^h VT^l$  a word of length  $h+l+1 \leq k$  in letters  $R, T, V$  where  $R^h$  means  $h$  consecutive letters equal to  $R$  and  $T^l$  means  $l$  consecutive letters equal to  $T$ . Then the set  $\mathcal{C}_{R^h VT^l}$  of points  $q \in \mathcal{C}^{h+l+1}(m)$  whose **RVT** code is  $R^h VT^l$  is an analytic submanifold of  $\mathcal{C}^{h+l+1}(m)$  of codimension  $l+1$ . Moreover, the fiber of the projection of  $\mathcal{C}_{R^h VT^l}$  onto  $(\mathcal{C}^h(m))_C$  over  $q_h \in (\mathcal{C}^h(m))_C$  is the set  $F^{h+l+1}(\hat{P}^l(\hat{S}(q_h)))$*
- (3) *Consider the the flag (3) associated to  $\mathcal{D}_{h+l+1}$ . The set  $\mathcal{C}_{R^h VT^l}$  can be defined inductively in the following way:*
- (i)  $\mathcal{C}_{R^h}$  the open set of points  $q$  such that  $q_h \in \mathcal{C}^h(m)$  is a Cartan point;
  - (ii)  $\mathcal{C}_{R^h V}$  the set of points  $q \in \mathcal{C}_{R^h}$  such that  $[\mathcal{D}_k]_{h+1}$  contains  $L[\mathcal{D}]_{h-1}$  at  $q$ ;
  - (iii)  $\mathcal{C}_{R^h VT^i}$  the set of points  $q \in \mathcal{C}_{R^h VT^{i-1}}$  such that  $[\mathcal{D}_k]_{h+i+1}$  is tangent to  $\mathcal{C}_{R^i V}$  (resp.  $\mathcal{C}_{R^i VT^{i-1}}$ ) at  $q$  for  $i = 1$  (resp.  $1 \leq i \leq l$ ).
- (4) *To a word  $R^h VT^l$  we associate a family  $\{K_0, \dots, K_l\}$  of directions in  $\mathbb{R}^{m+1}$  in the following way:*
- $K_i(q)$  is the direction generated by  $x_{h+i} - x_{h-1}$  for  $i = 0, \dots, l$
  - Let be  $(M_0, \dots, M_{h+l+1})$  an articulated arm in  $\mathbb{R}^{m+1}$ . Its configuration  $q = (x_0, \dots, x_{h+l+1})$  belongs to  $\mathcal{C}_{R^h VT^l}$  if and only if, in configuration  $q \in \mathcal{C}^k(m)$ , each segment  $[M_{i-1}, M_i]$  and  $[M_i, M_{i+1}]$  are not orthogonal at point  $M_i$  for each  $i = 1, \dots, h-1$  and each segment  $[M_{h+i}, M_{h+i+1}]$  is contained in the affine hyperplane though  $x_{h+i}$  which is orthogonal to the direction  $K_i$  for all  $i = 0, \dots, l$

##### Remark 4.5.

- (1) *The result of part (3) condition (iii) can be seen as a justification of term "tangency" as in the context [11] for  $m = 1$ .*
- (2) *For  $m = 1$ , in the context of the "car with  $n$  trailers", in [3], F. Jean has build a family of "critical angulars". In our situation ( $m \geq 2$ ) the family  $\{K_0, \dots, K_l\}$  of directions in  $\mathbb{R}^{m+1}$ , is nothing but a generalization of Jean's result.*
- (3) *A comparable result of part (3) and (4) can be find in [14] (with quite different assumptions), by using hyperspherical coordinates.*

*Proof.* fix some tangency point  $q \in \mathcal{C}^k(m)$ . From Remark 4.2 part (3), there must exist a vertical point  $q_i$  under  $q$ . Let  $q_l$  be the last such tangency point under  $q$ . Then we have  $\mathcal{A}_{l-1}(q) = 0$ . In fact , for  $q_{l-1} = (x_0, \dots, x_{l-1})$  given, this relation characterizes such points  $q_l = (x_0, \dots, x_{l-1}, x_l) \in \mathcal{C}^l(m)$  which are vertical. In other words, the set  $F^l(\hat{S}^l(q_{l-1}))$  are exactly the set of points  $(q_{l-1}, x_l) \in \mathcal{C}^l(m)$  such that  $\mathcal{A}_{l-1}(q_{l-1}, x_l) = 0$ .

Now, from the definition of  $l$  if  $l+1 \leq k$ ,  $q_{l+1}$  cannot be vertical. The point  $q_{l+1}$  is no more regular otherwise between  $q_{l+1}$  and  $q$  we must have a vertical point which contradicts the definition of  $l$ . So  $q_{l+1}$

must be a tangency point. Set  $q_{l+1} = (x_0, \dots, x_l, x_{l+1})$ .

Taking in account the proof of Proposition 3.1, in the trivialization  $\hat{\Psi}^l : \mathcal{D}_l \rightarrow \mathcal{C}^l(m) \times \mathbb{R}^{m+1}$  we have

$$\hat{\Psi}^l \left[ \sum_{r=1}^{m+1} (x_{l+1}^r - x_l^r)(x_l^r - x_{l-1}^r) Y_l + (x_{l+1}^r - x_l^r) \frac{\partial}{\partial x_l^r} \right] = x_{l+1} - x_l$$

Note that in fact  $\left[ \sum_{r=1}^{m+1} (x_{l+1}^r - x_l^r)(x_l^r - x_{l-1}^r) Y_l + (x_{l+1}^r - x_l^r) \frac{\partial}{\partial x_l^r} \right]$  is nothing but  $Y_{l+1}$  (see Remark 3.1). On

the other hand, on  $S(\mathcal{D}_l, \mathcal{C}^l, \gamma_l)$ , the vector field  $\left[ \sum_{r=1}^{m+1} (x_{l+1}^r - x_l^r)(x_l^r - x_{l-1}^r) Y_l + Z_l \right]$  belongs to  $[\mathcal{D}_l]^{[1]}$ . As  $q_{l+1}$

is a tangency point, this vector field must project on  $\mathcal{C}^l(m)$  onto a vector which is tangent to  $F^{l-1}(\hat{S}^l(q_{l-1}))$  at point  $q_l$ . So we must have:

$$(29) \quad D_{q_l} \mathcal{A}_{l-1}(Y_{l+1}) = 0.$$

Taking in account that  $\mathcal{A}_{l-1}(q_l) = 0$ , and Remark 3.1 we get  $Y_l = Z_{l-1}$  on  $\mathcal{A}_{l-1} = 0$ . According to the value of  $Z_{l-1}$  and  $Z_l$ , modulo  $\mathcal{A}_{l-1} = 0$ , equation (29) gives rise to the relation

$$(30) \quad < x_{l+1} - x_l, (x_l - x_{l-1}) + (x_{l-1} - x_{l-2}) > = < x_{l+1} - x_l, x_l - x_{l-2} > = 0$$

Note that, for  $q_{l-1} = (x_0, \dots, x_l)$  fixed, the equation  $\mathcal{A}_{l-1}(q_l) = 0$  associated to (30) are exactly the equations of the set  $F^{l+1}(\hat{P}^1(\hat{S}^l(q_{l-1})))$ . Moreover, it is easy to see that these equations are independent on  $\mathcal{C}^{h+2}(m)$ . Assume that for  $l \leq i < k$  the point  $q_i$  is tangency. By same arguments as previously,  $q_{i+1}$  must also a tangency point. So, by induction we get part (1).

We now look for part (2). Note that, from the previous proof we have already shown that  $\mathcal{C}_{R^h V T}$  is an analytic submanifold of  $\mathcal{C}^{h+2}(m)$  of codimension 2 and that the fiber of the projection of  $\mathcal{C}_{R^h V T}$  onto  $\mathcal{C}_C^h(m)$  over  $q_h \in \mathcal{C}_C^h(m)$  is the set  $F^{h+2}(\hat{P}^1(\hat{S}(q_h))) \cap \mathcal{C}^{h+2}(m)$ . The global result will be obtained by induction on  $l$ .

To precise the induction hypothesis we need the following notations and Lemma.

We set  $\mathcal{A}_{i,j}(q) = < x_{i+1} - x_i, x_{j+1} - x_j >$  for  $i, j = 0, \dots, k-1$ . The family of functions  $\{\mathcal{A}_{i,j}\}$  have the immediate following properties:

$$\mathcal{A}_{i,j} = \mathcal{A}_{j,i}, \quad \mathcal{A}_{i,i} = 1, \quad \mathcal{A}_{i,i-1} = \mathcal{A}_i$$

**Lemma 4.2.** *For any  $i > j$  we have*

- (1)  $D\mathcal{A}_{i,j}(Z_h) = 0$  for all  $0 \leq h \leq k-1$  and  $h \neq i, i+1, j, j+1$  ;
- (2)  $D\mathcal{A}_{i,j}(Z_i) = -\mathcal{A}_{i,j}$  and  $j \neq i-1$
- (3)  $D\mathcal{A}_{i,i-1}(Z_i) = 1 - \mathcal{A}_{i,i-1}$
- (4)  $D\mathcal{A}_{i,j}(Z_{i+1}) = \mathcal{A}_{i+1,j}$
- (5)  $D\mathcal{A}_{i,j}(Z_{j+1}) = \mathcal{A}_{i,j+1}$

*Proof of Lemma.*

At first note that  $Z_h$  is a vector field with components only on  $\{\frac{\partial}{\partial x_h^1}, \dots, \frac{\partial}{\partial x_h^{m+1}}\}$  so part (1) is then a consequence of the definition of  $\mathcal{A}_{i,j}$ . The other parts is a direct calculation.  $\square$

*Induction hypothesis ( $H_i$ ):* Assume that for any  $1 \leq i < l$  we have:

- each set  $\mathcal{C}_{R^h V T^i}$  of point  $q \in \mathcal{C}^{h+i+1}(m)$  whose **RVT** code is  $R^h V T^i$  is a subset of  $\mathcal{C}^{h+i+1}(m)$  which is defined by the system of independent equations of type

$$(31) \quad \begin{cases} \phi_0 = \mathcal{A}_h = 0 \\ \phi_1 = D_{q_{h+1}} \phi_0(Y_{h+2}) = 0 \\ \phi_2 = D_{q_{h+2}} \phi_1(Y_{h+3}) = 0 \\ \dots \quad \dots \\ \phi_i = D_{q_{h+i}} \phi_{i-1}(Y_{h+i+1}) = 0 \end{cases}$$

- for  $j = 0, \dots, n$  modulo the previous equations  $\phi_{j'} = 0$  for  $0 \leq j' < j$ , the equation  $\phi_j = 0$  can be reduced to

$$(32) \quad \phi_j \equiv \bar{\phi}_j = \sum_{i'=h-1}^{h+j-1} \mathcal{A}_{h+j,i'} = 0$$

• the set  $F^{h+i+1}(\hat{P}^i(\hat{S}(q_h)) \cap \mathcal{C}^{h+i+1}(m)$  is defined by system of equations (31), but for each fixed  $q_h \in \mathcal{C}^h(m)_C$

• we have  $D\bar{\phi}_j(Y_{h+i+1}) = 0$  for all  $0 \leq j < i$

Note that we have already shown that this hypothesis is true for  $i = 1$ .

Referring to the previous proof for point  $q_{i+1}$  to obtain the equation (29), from the same argument applied for  $q_{h+i+1} = (q_{h+i}, x_{h+i+1})$ , we get that, when  $q_{h+i} \in \mathcal{C}_{R^h VT^i}$ , the point  $q_{h+i+1}$  belongs to  $\mathcal{C}_{R^h VT^{i+1}}$  if and only if

$$D_{q_{h+i}} \phi_j(Y_{h+i+2}) = 0$$

for all  $0 \leq j < i+1$ .

According to (32), these conditions are equivalent to

$$D_{q_{h+i}} \bar{\phi}_j(Y_{h+i+2}) = 0$$

for all  $0 \leq j < i+1$ .

On one hand, the vector field  $Y_{h+i+2}$  can be written

$$Y_{h+i+2} = \mathcal{A}_{h+i+1} Y_{h+i+1} + \mathcal{Z}_{h+i+1}$$

and so, from our induction hypothesis, on  $\mathcal{C}_{R^h VT^i}$ , we have

$$D_{q_{h+i}} \bar{\phi}_j(Y_{h+i+1}) = 0$$

for all  $0 \leq j < i$ . On the other hand, from (32), each  $\bar{\phi}_j$ , depends only on variables  $(x_{h-1}, \dots, x_{h+j+1})$ , so we always have  $D\phi_j(\mathcal{Z}_{h+i+1}) = 0$  for all  $0 \leq j < i$ . So, the only equation we must add to the system (31) for  $j = i$  is:

$$(33) \quad D_{q_{h+i}} \bar{\phi}_i(Y_{h+i+2}) = 0$$

In fact, it is clear that to the system (31), this equation is of course equivalent to

$$\phi_{i+1} = D_{q_{h+i}} \phi_i(Y_{h+i+2}) = 0$$

At first, note that, for  $q_h \in \mathcal{C}_C^h(m)$  given, the system (31) with adding the equation  $\phi_{i+1} = 0$  characterizes any point  $q_{h+i+1} = (q_{h+i}, x_{h+i+1})$  which belongs to  $F^{h+i+2}(\hat{P}^{i+1}(\hat{S}(q_h)))$ . We must show now that  $\phi_{i+1}$  has a reduction of type (32), modulo the previous equation

From our hypothesis that  $\mathcal{A}_h = 0$  and Remark 3.1: we have

$$Y_{h+i+2} = [\sum_{i'=h}^{h+i} \prod_{j'=i'+1}^{h+i+1} \mathcal{A}_{j'} \mathcal{Z}_{i'}] + \mathcal{Z}_{h+i+1}$$

Using the decomposition (32) and Lemma 4.2, we compute  $D\mathcal{A}_{k+j,i''}(\tilde{Z}_{i'})$  for  $i'' = h-1, \dots, h+j-1$  and  $i' = h, \dots, h+j+1$ :

• for  $i' = h, \dots, h-j+1$  fixed, we have

$$\begin{aligned} D\mathcal{A}_{h+j,i''}(\tilde{Z}_{i'}) &= \mathcal{A}_{h+j,i'} \text{ for } i'' = i' - 1 \\ &= -\mathcal{A}_{h+j,i'} \text{ for } i'' = i' \\ &= 0 \text{ for } i'' \neq i', i' - 1 \end{aligned}$$

• for  $i' = h+j$ , we have:

$$\begin{aligned} D\mathcal{A}_{h+j,i''}(\tilde{Z}_{i'}) &= \mathcal{A}_{h+j=1,i''} \text{ for } i'' \neq h+j-1 \\ &= 1 - \mathcal{A}_{h+j+1,h+j-1} \text{ for } i'' = h+j-1 \end{aligned}$$

• for  $i' = h+j+1$ , we have:

$$\begin{aligned} D\mathcal{A}_{h+j,i''}(\tilde{Z}_{i'}) &= \mathcal{A}_{h+j=1,i''} \text{ for } i'' \neq h+j-1 \\ &= 1 - \mathcal{A}_{h+j+1,h+j-1} \text{ for } i'' = h+j-1 \end{aligned}$$

So, we finally get:

$$D\bar{\phi}_i(Y_{h+i+2}) = -\mathcal{A}_{h+j+1} \bar{\phi}_i + \sum_{i'=h-1}^{h+j} \mathcal{A}_{h+j+1,i'}$$

of course we set  $\bar{\phi}_{i+1} = \sum_{i'=h-1}^{h+j} \mathcal{A}_{h+j+1,i'}$ .

We must also show that  $\{\phi_0, \phi_1, \dots, \phi_i, \phi_{i+1}\}$  is a set of independent functions. Using the reductions of equation (31), it is sufficient to prove that  $\{\bar{\phi}_0, \bar{\phi}_1, \dots, \bar{\phi}_i, \bar{\phi}_{i+1}\}$  is a set of independent functions

From the expression of  $\bar{\phi}_j$ ,  $j = 0, \dots, i+1$  ( of type (32)), we have

$$\bar{\phi}_j = \langle x_{h+j+1} - x_{h+j}, \sum_{i'=h-1}^{h+j} (x_{i'+1} - x_{i'}) \rangle = x_{h+j+1} - x_{h+j}, x_{h+j} - x_{h-1} \rangle$$

So we have

$$D\bar{\phi}_j \left( \frac{\partial}{\partial x_{h+j+1}^r} + \frac{\partial}{\partial x_{h+j}^r} \right) = x_{h+j+1} - x_{h+j}$$

So  $D\bar{\phi}_j \left( \frac{\partial}{\partial x_{h+j+1}^r} + \frac{\partial}{\partial x_{h+j}^r} \right) \neq 0$  for some  $1 \leq r \leq m+1$  on  $\mathcal{C}^{h+i+2}(m)$ .

It follows that  $\{\bar{\phi}_0, \bar{\phi}_1, \dots, \bar{\phi}_i, \bar{\phi}_{i+1}\}$  is a set of independent functions on  $\mathcal{C}^{h+i+2}(m)$ .

This ends the proof of part (2)

Now, in part (3), the condition (i) and (ii) are a direct consequence of Proposition 4.1. We will look for condition (iii).

For simplicity, we denote only by  $D$  the distribution  $\mathcal{D}_{h+l+1}$  and consider the flag associated to  $\mathcal{D}$ :

$$D = D_{h+l+1} \subset D_{h+l} \subset \dots \subset D_j \subset \dots \subset D_1 \subset D_0 = T\mathcal{C}^{h+l+1}(m)$$

For each  $0 \leq i < l$ , we can identify  $\mathcal{C}^{h+i+1}(m)$  with the submanifold of  $\mathbb{R}^{h+l+1}$  of equations:

$$\begin{cases} x_{h+j+2} = 0 & \text{for } j = i+1, \dots, l \\ \|x_j - x_{j-1}\| = 1 & \text{for } j = 1, \dots, h+i+1 \end{cases}$$

With these identification, we have:

**Lemma 4.3.** : each member  $D_j$  of the previous flag is equal to  $D_{j-1} + \mathcal{D}_j$  for each  $j = h+l+1, h+l, \dots, 0$

*Proof of Lemma 4.3.* For  $j = h+l+1$  the result is trivial. Assume that the result is true for  $D_j$  for some  $0 < j \leq h+l+1$ . We know that  $D_{j-1}$  is a distribution of rank  $(h+l-j)m+1$  which contained  $D_j$  and is generated by Lie brackets  $[X, Y]$  for all vector fields  $X$  and  $Y$  tangents to  $D_j$ . We first prove that  $D_j + \mathcal{D}_{j-1}$  is contained in  $D_{j-1}$ . We know that  $\mathcal{D}_j$  is generated by the family  $\{(x_j^r - x_{j-1}^r)y_j + \frac{\partial}{\partial x_j^r} : r = 1, \dots, m+1\}$  and  $Y_j = \mathcal{A}_{j-1}Y_{j-1} + \mathcal{Z}_{j-1}$  also belongs to  $\mathcal{D}_j$  (Remark 3.1). As  $Y_{j-1}$  depends only on variables  $(x_0, \dots, x_{j-1})$  and is tangent to  $\mathcal{C}^{h+j}(m)$ , we have then

$$\begin{aligned} [(x_j^r - x_{j-1}^r)Y_j + \frac{\partial}{\partial x_j^r}, Y_j] &= -Y_j\{(x_j^r - x_{j-1}^r)\}Y_j + [\frac{\partial}{\partial x_j^r}, Y_j] \\ &= (x_j^r - x_{j-1}^r)Y_j + \frac{\partial \mathcal{A}_{j-1}}{\partial x_j^r} Y_{j-1} + [\frac{\partial}{\partial x_j^r}, \mathcal{Z}_{j-1}] \\ &= (x_j^r - x_{j-1}^r)Y_j + (x_{j-1}^r - x_{j-2}^r)Y_{j-1} + \frac{\partial}{\partial x_{j-1}^r} \text{ for } j \geq 2 \\ &= (x_1^r - x_0^r)Y_1 + \frac{\partial}{\partial x_0^r} \text{ for } j = 0 \end{aligned}$$

So, we get  $D_j + \mathcal{D}_{j-1}$  is contained in  $D_{j-1}$ . On the other hand, on  $\mathcal{C}^{h+j}(m)$  the distribution generated by  $D_j$  and the family  $\{(x_j^r - x_{j-1}^r)Y_j + \frac{\partial}{\partial x_j^r} : r = 1, \dots, m+1\}$  is equal to the distribution generated by  $D_j$

and  $\{\frac{\partial}{\partial x_j^r} : r = 1, \dots, m+1\}$  which is again  $D_j + \mathcal{D}_{j-1}$ . But, according to the constraint  $\|x_j - x_{j-1}\| = 1$  in  $\mathcal{C}^{h+j}(m)$ , we get that  $D_j + \mathcal{D}_{j-1}$  is a distribution of constant rank  $(h+l-j)m+1$  which ends the proof.  $\square$

We come back to the proof of condition (iii) of part (2). According to the fact that (i), (ii) are already proved, assume that condition (iii) is true for all  $0 \leq i \leq j < l$ .

From this assumption we already know that  $D_{h+j+1}$  is tangent to  $\mathcal{C}_{R^iVT^j}$  at each point of this manifold. So, according to Lemma 4.3, the set of points  $q = (x_0, \dots, x_{h+l+1}) \in \mathcal{C}^{h+l+1}(m)$  such that  $D_{h+j+2}$  is tangent to  $\mathcal{C}_{R^iVT^j}$  is the set of points where  $D_{h+j+2}$  is tangent to  $\mathcal{C}_{R^iVT^j}$ . But, we have already shown that  $\mathcal{C}_{R^iVT^j}$  is defined by the system (31), for  $0 \leq j \leq l$ . Moreover this system of equations depends only on variables  $(x_{h-1}, \dots, x_{h+j+1})$ . As  $D_{h+j+2}$  is generated by  $\{Y_{h+j+2} + \frac{\partial}{\partial x_{h+j+2}}\}$ , this last condition is reduced to

$$D\phi_i(Y_{h+j+2}) = 0 \text{ for all equations } \phi_i = 0 \text{ in (31).}$$

From the value of  $Y_{h-j+2}$  and the properties of these equations, the previous condition is equivalent to:

$$\bar{\phi}_{j+1} \equiv D\bar{\phi}_j(Y_{h+j+2}) = 0.$$

So finally, the set of points  $q = (x_0, \dots, x_{h+l+1}) \in \mathcal{C}^{h+l+1}(m)$  such that  $D_{h+j+2}$  is tangent to  $\mathcal{C}_{R^iVT^j}$  is the set defined by the system (31). This ends the proof of condition (iii) of part (3).

The part (4) we have already noted that

$$(34) \quad \sum_{i'=h-1}^{h+j-1} \mathcal{A}_{h+j,i'} < x_{h+j+1} - x_{h+j}, x_{h+j} - x_{h-1} >$$

So part (4) is a simple geometrical interpretation of reduced system  $\{\bar{\phi}_j = 0, j = 0, \dots, l\}$  of the set  $\mathcal{C}_{R^iVT^l}$   $\square$

**Remark 4.6.**

Let be  $q \in \mathcal{C}^k(m)$  and assume that its projection  $q_{h+1}$  is vertical. For any  $0 \leq l \leq k - h - 1$ , in the set  $F^{h+l+1}(\hat{P}^l(\hat{S}(q_h)))$ , consider the distribution  $F_*^{h+l+1}(\mathfrak{d}_i^k)$ . Then  $F_*^{h+l+1}(\mathfrak{d}_i^k)(q_{h+1+l})$  is a critical hyperplane in  $\mathcal{D}_{h+1+l}(q_{h+1+l})$ . Then, according to Theorem 4.3 part (2) and part (4) and (34), on  $\mathcal{C}^{h+l+1}(m)$ , a point  $z \in \mathcal{D}_{h+1+l}(q_{h+1+l})$  belongs to  $F_*^{h+l+1}(\mathfrak{d}_i^k)(q_{h+1+l})$  if and only if  $z$  is orthogonal to  $x_{h+l} - x_{h-1}$ .

## 5. RELATION BETWEEN EKR CLASSES OF AT MOST DEPTH 1, **RVT** CODES AND ARTICULATED ARMS

### 5.1. Mormul EKR classes according to [7] and [8].

In [7] and [8], P. Mormul has constructed a coding system for labeling singularity classes of germs of special multi-flag which he called "Extended Kumpera-Ruiz classes" ("EKR classes" in short). Mormul's codes are finite sequences in  $\mathbb{N}$ . For each such a sequence (under some constraints), using Theorem 2.3, he build an associated polynomially germ at  $\mathbf{0} \in \mathbb{R}^N$ , for  $N$  large enough, of  $(k+1)$ -dimensional distributions. These germs are called "Extended Kumpera-Ruiz pseudo-normal forms".

#### Notations 5.1.

$\mathbb{R}^s(y_1, \dots, y_s)$  is the space  $\mathbb{R}^s$  provided with coordinates  $(y_1, \dots, y_s)$  in the neighbourhood of 0.

In the following construction, any external variables  $x$  will denoted by a capital letter  $X = x + c$ , when such a variable can be shifted by a constant  $c$ , not excluding, the value  $c = 0$  and only by  $x$  when such a shift is excluded.

For each  $j \in \{1, 2, \dots, m+1\}$ , we will define an operation denoted  $\mathbf{j}$  producing new  $(m+1)$ -distributions from the older ones. More precisely, given a distribution generated on a neighbourhood of  $0 \in \mathbb{R}^s(y_1, \dots, y_s)$  the operation  $\mathbf{j}$ , as operation at level  $l$ , gives rise to a new  $(m+1)$ -distribution defined in the neighbourhood of 0 some space  $\mathbb{R}^{s+m}(y_1, \dots, y_s, x_1^l, \dots, x_m^l)$ , generated by the vector fields

$$Z'_1 = x_1^l Z_1 + \dots + x_{j-1}^l Z_{j-1} + Z_j + X_j^l Z_{j+1} + \dots + X_m^l Z_{m+1}$$

and by  $Z'_2 = \frac{\partial}{\partial x_1^l}, \dots, Z'_{m+1} = \frac{\partial}{\partial x_m^l}$ .

It is important that these new local generators are written precisely in this order, thus producing a new system of generators  $(Z'_1, \dots, Z'_{m+1})$ .

For instance, when  $m = 2$ , we have the three following possibility for  $\mathbf{j}$ :

$$Z'_1 = \begin{cases} Z_1 + X_1^l Z_2 + X_2^l Z_3 & \text{for } \mathbf{j} = \mathbf{1} \\ Z_1 + x_1^l Z_2 + X_2^l Z_3 & \text{for } \mathbf{j} = \mathbf{2} \\ Z_1 + x_1^l Z_2 + x_2^l Z_3 & \text{for } \mathbf{j} = \mathbf{3} \end{cases}$$

Extended Kumpera-Ruiz pseudo-normal forms (EKR), of length  $k \geq 1$ , denoted by  $\mathbf{j}_1 \mathbf{j}_2 \dots \mathbf{j}_k$ , where  $j_1, j_2, \dots, j_k \in \{1, 2, \dots, m+1\}$ , are defined inductively, starting from the distribution generated by

$$\left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x_1^0}, \dots, \frac{\partial}{\partial x_k^0} \right)$$

in the neighbourhood of  $0 \in \mathbb{R}^{k+1}(t, x_1^0, \dots, x_k^0)$ , which is the tangent bundle of  $\mathbb{R}^{k+1}(t, x_1^0, \dots, x_k^0)$ .

Then, if we assume that the germ associated to  $\mathbf{j}_1 \dots \mathbf{j}_{r-1}$  is already known, the new germ associated to  $\mathbf{j}_1 \dots \mathbf{j}_r$  is the result of the operation  $\mathbf{j}_r$  performed, as the operation at level  $r$ , from the germ associated to  $\mathbf{j}_1 \dots \mathbf{j}_{r-1}$ . Such a differential system will be denoted by  $\Delta_{\mathbf{j}_1 \mathbf{j}_2 \dots \mathbf{j}_r}$ .

We have then the following result:

**Theorem 5.1.** [7] [8] *Let  $D$  differential system which generates a special multi-flag of step  $m \geq 2$  and length  $k \geq 1$ , on a manifold  $M$  of dimension  $(k+1)m+1$ . In a neighbourhood of every point  $p \in M$ ,  $(M, D, p)$  is locally equivalent to some differential system  $(\Delta_{\mathbf{j}_1 \mathbf{j}_2 \dots \mathbf{j}_k}, \mathbb{R}^{(k+1)m+1}, 0)$ . Moreover, that  $\Delta_{\mathbf{j}_1 \mathbf{j}_2 \dots \mathbf{j}_k}$  can be taken such that  $\mathbf{j}_1 = \mathbf{1}$  and, for  $l = 1 \dots, k-1$ , if  $\mathbf{j}_{l+1} > \max(\mathbf{j}_1, \mathbf{j}_2, \dots, \mathbf{j}_l)$ , then  $\mathbf{j}_{l+1} = 1 + \max(\mathbf{j}_1, \mathbf{j}_2, \dots, \mathbf{j}_l)$  (the rule of the least possible new jumps upwards in the words  $\mathbf{j}_1 \mathbf{j}_2 \dots \mathbf{j}_k$ ).*

Note, however, that possible constants in such an EKR representing a given germ  $D$  are not, in general, uniquely defined. On the other hand, for a given germ of distribution  $D$ , the sequence of operations  $j_1 j_2 \dots j_r$  associated to it, is unique and satisfy the rule of least upward jumps. Such a sequence is called a **EKR class of multi-flags** and the integer  $d = \sup\{j_1, \dots, j_k\} - 1$  is called the **depth** of this EKR class.

## 5.2. Stratification of EKR classes of at most 1-depth by RVT codes.

We denote by  $\Sigma_{j_1 \dots j_k}$  the set of points  $q \in \mathcal{C}^k(m)$  such that germ of distribution at  $q$  belongs to the EKR class  $j_1, \dots, j_k$ . Recall in a word  $\omega$ , a sub-word of type  $R^h$  or  $T^l$  means a sequence of  $h$  (resp.  $l$ ) consecutive letters  $R$  (resp.  $T$ ) if  $h > 0$  (resp.  $l > 0$ ), and no letter  $R$  (resp.  $T$ ) if  $h = 0$  (resp.  $l = 0$ ). For any EKR class of 1-depth we will denote by  $\{i_1, \dots, i_\nu\}$  the set  $\{i \text{ such that } j_i = 2\}$ .

The following result gives a complete description of EKR classes of at most 1-depth in terms of **RVT** classes. In particular, it contains the results announced in Theorem 2 and part (1) of Theorem 3:

### Theorem 5.2.

- (1)  $\Sigma_{1 \dots 1}$  is of Cartan points. So  $\Sigma_{1 \dots 1}$  is an open dense set whose complementary is a subanalytic set of  $\mathcal{C}^k(m)$  of codimension 1
- (2) for any EKR class  $j_1, \dots, j_k$  of 1-depth, the set  $\Sigma_{j_1 \dots j_k}$  is an analytic submanifold of  $\mathcal{C}^k(m)$  of codimension equal to  $\nu$  (cardinal of index  $i$  such that  $j_i = 2$ ). Moreover,  $q$  belongs to  $\Sigma_{j_1 \dots j_k}$  if and only if the articulated arm  $M_0, \dots, M_k$  the segments  $[M_{i-2}, M_{i-1}]$  and  $[M_{i-1}, M_i]$  are orthogonal in  $M_{i-1}$  for all  $i = i_1, \dots, i_\nu$  and any other pair of consecutive segments are not orthogonal
- (3) in the previous situation we have:
  - (i) a point  $q$  belongs to  $\Sigma_{j_1 \dots j_k}$  if and only if its **RVT** code is a word such that the only letters  $V$  are at rank  $i_1, \dots, i_\nu$ .
  - (ii) A **RVT** class  $\mathcal{C}_\omega$  is contained in  $\Sigma_{j_1 \dots j_k}$ , if and only if  $\omega$  is of type  $R^{h_0} V T^{l_1} R^{h_1} \dots V T^{l_\nu} R^{h_\nu}$  and each letter  $V$  are exactly at rank  $i_1, \dots, i_\nu$ . More precisely we have  $l_\lambda + h_\lambda = i_{\lambda+1} - i_\lambda - 1$  for  $\lambda = 1, \dots, \nu-1$  and  $l_\nu + h_\nu = k - i_\nu - 1$ . Such a class is an analytic submanifold of  $\Sigma_{j_1 \dots j_k}$  of codimension  $l_1 + \dots + l_\nu$ . In particular  $\mathcal{C}_{R^{h_0} V R^{h_1} \dots V R^{h_\nu}}$  is an open dense set of  $\Sigma_{j_1 \dots j_k}$  for  $h_0 = i_1 - 1$ ,  $h_\lambda = i_{\lambda+1} - i_\lambda - 1$  for  $\lambda = 1, \dots, \nu-1$  and  $h_\nu = k - i_\nu - 1$ .
  - (iii)  $\Sigma_{j_1 \dots j_k}$  is the union of all classes of type  $\mathcal{C}_{R^{h_0} V T^{l_1} R^{h_1} \dots V T^{l_\nu} R^{h_\nu}}$  which satisfies the previous properties (ii)

### Remark 5.1.

The decomposition of  $\Sigma_{j_1 \dots j_k}$  given in (iii) into **RVT** classes is in agreement with the decomposition of such EKR classes for  $k = 3$  described by Howard in appendix of [4]. So the description in (iii) can be seen as a generalization of Howard's result.

For the proof of this Theorem we need the following Lemma:

### Lemma 5.1.

given an EKR class  $\Sigma_{j_1 \dots j_k}$  of at most 1-depth, then the index  $j_l = 2$ , for  $l > 1$ , if and only if  $\mathcal{A}_{l-1} = 0$

*Proof.* From the definition of the "operation  $\mathbf{j}$ ", it follows that a point  $q_l$  is regular if and only if  $j_l = 1$ . Now from the context of the proof Lemma 4.1 it follows that  $q_l$  is regular if and if  $\mathcal{A}_{l-1} \neq 0$ . Now, as EKR class  $\Sigma_{j_1 \dots j_k}$  of at most 1-depth it follows that  $j_l = 2$  if and only if  $\mathcal{A}_{l-1} = 0$ . □

### Proof of Theorem 5.2.

The parts (1) and (2) are consequences of Lemma 5.1, Proposition 4.1 and Theorem 4.2 part (1). For part (3), the property (i) is also a consequence of are consequences of Lemma 5.1 and Proposition 4.1.

We now look for property (ii). If  $\omega$  is of type  $R^{h_0} V T^{l_1} R^{h_1} \dots V T^{l_\nu} R^{h_\nu}$  and each letter  $V$  are exactly at rank  $i_1, \dots, i_\nu$ , from part (i),  $\mathcal{C}_{R^{h_0} V T^{l_1} R^{h_1} \dots V T^{l_\nu} R^{h_\nu}}$  must be contained in  $\Sigma_{j_1 \dots j_k}$ . On the other hand consider any word  $\omega$  in **RVT** code such that each letters of rank  $r_1, \dots, r_\nu$  is  $V$  and take any  $q \in \mathcal{C}_\omega \subset \mathcal{C}^k(m)$ . If  $i_2 - i_1 > 0$ , consider some  $q_l$  under  $q$  with  $i_1 < l < i_2$ . Assume that  $q_l$  is critical. According to the **RVT** code of  $q$ , the point  $q_l$  must be tangency from Theorem 4.3 part (1), and, moreover, for  $i_1 < l' \leq l$ , the point

$q_{l'}$  must be also tangency. Otherwise  $q_l$  must be regular, and then, according to the **RVT** code of  $q$  and Theorem 4.3 part (1), each point  $q_{l'}$  must be also regular for  $l \leq l' < i_2$ . It follows that the **RVT** code of  $q_{i_2-1}$  is of type  $R^{h_0}VT^{l_1}R^{h_1}$ . By induction on  $1 \leq i \leq \nu$ , using the same arguments we get that  $\omega$  must be of type  $R^{h_0}VT^{l_1}R^{h_1} \dots VT^{l_\nu}R^{h_\nu}$ .

Finally, from the proof Theorem 4.2 part (2), it follows that the equations of  $\mathcal{C}_{R^{h_0}VT^{l_1}R^{h_1} \dots VT^{l_\nu}R^{h_\nu}}$  is the union of  $\nu$  systems of type (31) and so gives rise to  $\nu + l_1 + \dots + l_\nu$  independent equations. So  $\mathcal{C}_{R^{h_0}VT^{l_1}R^{h_1} \dots VT^{l_\nu}R^{h_\nu}}$  is an analytic submanifold of  $\mathcal{C}^k(m)$  of codimension  $\nu + l_1 + \dots + l_\nu$ . On the other hand, the equations of  $\Sigma_{j_1 \dots j_k}$  are  $\mathcal{A}_{i_\lambda-1} = 0$  for  $\lambda = 1, \dots, \nu$ . These equations are exactly the first equations of the  $\nu$  systems of type (31) which define  $\mathcal{C}_{R^{h_0}VT^{l_1}R^{h_1} \dots VT^{l_\nu}R^{h_\nu}}$ . This ends the proof of the property (ii).

The property (iii) is a direct consequence of properties (i) and (ii).  $\square$

### 5.3. EKR classes of 1-depth , RVT codes and configurations of articulated arms.

We will now give a complete interpretation of the previous result in terms of configuration of an articulated arm such is announced in Theorem 3 part (2):

#### Theorem 5.3.

Let be  $\Sigma_{j_1 \dots j_k}$  a EKR class of depth 1 and  $\{i_1, \dots, i_\nu\}$  the set  $\{i \text{ such that } j_i = 2\}$ . There exists a family

$$\{K_0^\lambda, \dots, K_{\varkappa_\lambda}^\lambda\}_{\lambda=1, \dots, \nu}, \text{ with } \varkappa_\lambda = i_{\lambda+1} - i_\lambda - 1 \text{ for } \lambda = 1, \dots, \nu - 1 \text{ and } \varkappa_\nu = k - i_\nu - 1$$

of directions in  $\mathbb{R}^{m+1}$  such that  $q$  belongs to the class  $\mathcal{C}_{R^{h_0}VT^{l_1}R^{h_1} \dots VT^{l_\nu}R^{h_\nu}} \subset \Sigma_{j_1 \dots j_k}$  if and only if in configuration  $q$  we have:

- each segment  $[M_{i_\lambda+l-1}, M_{i_\lambda+l}]$  is orthogonal to  $K_l^\lambda$  in  $M_{i_\lambda+l-1}$ , for  $l = 0, \dots, \varkappa_\lambda$  and  $\lambda = 1, \dots, \nu$ ;
- each pair of consecutive segments  $[M_{i-2}, M_{i-1}]$  and  $[M_{i-1}, M_i]$  are not orthogonal in  $M_{i-1}$  for all  $i$  which do not belongs to the union of the previous sets of index.

#### Remark 5.2.

The property (ii) of Theorem 4.3 is of course a particular case of Theorem 5.2. Note that we can find an analogue result in [14] with a more restricted context.

For the proof of this result, we need the notion of "induced articulated arm" :

Given two integers  $r$  and  $s$  such that  $0 \leq r < s \leq k$ , we can look for the motion of an *induced articulated arm*, which consists of segments of the original articulated arm which joints  $M_r$  to  $M_s$  included. We can then study the *induced articulated arm*  $(M_r, \dots, M_s)$ . We put  $\kappa = s - r$ , and we denote by  $\mathcal{C}^{rs}(m)$  the image of  $\mathcal{C}^k(m)$  by the canonical projection  $\varrho^{rs}$  from  $\mathbb{R}_0^{m+1} \times \dots \times \mathbb{R}_k^{m+1}$  onto  $\mathbb{R}_r^{m+1} \times \dots \times \mathbb{R}_s^{m+1}$ .

In fact, we have:

$$\mathcal{C}^{rs}(m) = \{q_{rs} = (x_r, x_{r+1}, \dots, x_s)\}, \text{ if } q = (x_0, \dots, x_k)\}$$

Taking in account subsection 3.1, let be  $\mathcal{E}_{rs}$  is the distribution on  $(\mathbb{R}^{m+1})^{\kappa+1}$  generated by

$$\mathcal{Z}_r, \dots, \mathcal{Z}_{s-1}, \frac{\partial}{\partial x_s^1}, \dots, \frac{\partial}{\partial x_s^{m+1}}$$

and let be  $\mathcal{D}_{rs}$  the distribution induced by  $\mathcal{E}_{rs}$  on  $\mathcal{C}^{rs}(m)$ .

In terms of Notations 3.1, the mechanical system of the evolution of an induced arm  $(M_r, \dots, M_s)$ , is a controlled system on  $\mathbb{R}^{m+1} \times (\mathbb{S}^m)^\kappa \equiv \mathcal{C}^\kappa(m)$  naturally associated to the distribution  $\mathcal{D}_{rs}$ .

Consider a word  $R^{h_0}VT^{l_1}R^{h_1} \dots VT^{l_\nu}R^{h_\nu}$  of  $k$  letters in **RVT** code and we associate to this word the sequences  $r_0, \dots, r_\nu$  and  $s_0, \dots, s_\nu$  defined by:

$$s_0 = h_0 \text{ and } r_0 = 0;$$

$$s_i = s_{i-1} + h_i + l_i + 1 = h_0 + h_1 + l_1 + 1 + \dots + h_i + l_i + 1 \text{ and } r_i = s_{i-1} - 1 \text{ for } i = 1, \dots, \nu$$

We have the following characterization:

#### Lemma 5.2.

The configuration  $q \in \mathcal{C}^k(m)$  of an articulated arm  $(M_0, \dots, M_k)$  belongs to the class  $\mathcal{C}_{R^{h_0}VT^{l_1}R^{h_1} \dots VT^{l_\nu}R^{h_\nu}}$  if and only if the induced articulated arm associated to integer  $(r_i, s_i)$  satisfies  $\varrho^{r_i s_i}(q)$  belongs to the class  $\mathcal{C}_{R^{h_0}} \subset \mathcal{C}^{r_0 s_0}(m) = \mathcal{C}^{s_0}(m)$  for  $i = 0$  and to the class  $\mathcal{C}_{RVT^{h_i}R^{h_i}} \subset \mathcal{C}^{r_i s_i}(m)$  for all  $i = 1, \dots, \nu$



*Proof.*

For any  $q \in \mathcal{C}^k(m)$ , as usual we denote by  $q_l$  any point of  $\mathcal{C}^l(m)$  under  $q$ . Fix a configuration  $q \in \mathcal{C}^k(m)$  of the articulated arm  $(M_0, \dots, M_k)$ . At first for  $i = 0$ , the induced articulated arm associated  $(r_0, s_0)$  has the induced configuration  $q_{s_0}$ . The **RVT** code of  $q_{s_0}$  is the  $h_0$  first letters of the **RVT** code of  $q$ . So, these  $h_0$  first letters are  $R^{h_0}$  if and only if the **RVT** code of  $q_{s_0}$  is  $R^{h_0}$ .

Assume that we have proved that the  $s_i$  first letters of the **RVT** code of  $q$  are  $R^{h_0}VT^{l_1}R^{h_1} \dots VT^{l_i}R^{h_i}$  if and only if **RVT** code of the configuration  $q_{r_i s_i} = \varrho^{r_i s_i}(q)$  of the associated induced articulated arm is  $R^{h_0}$  for  $i = 0$  and  $RVT^{l_i}R^{h_i}$  for all  $1 \leq i \leq \mu - 1 < \nu$ .

Consider the configuration  $q_{r_\mu s_\mu} = \varrho^{r_\mu s_\mu}(q) \in \mathcal{C}^{r_\mu s_\mu}(m)$  of the associated induced articulated arm. Denote by  $q_{r_\mu l}$  the configuration under  $q_{r_\mu s_\mu}$  for  $l = r_\mu, \dots, s_\mu$  and we set  $\kappa_\mu = s_\mu - r_\mu - 1$ . By convention, the **RVT** code of  $q_{r_\mu r_\mu + 1}$  is  $R$ . Now, according to Proposition 4.1, in  $\mathcal{C}^{r_\mu s_\mu}(m) \equiv \mathcal{C}^{\kappa_\mu}(m)$ ,  $q_{r_\mu r_\mu + 2}$  is vertical if and only if

$$\sum_{j=1}^{m+1} (x_\mu^j - x_{\mu+1}^j)(x_{\mu+1}^j - x_{\mu+2}^j) = 0$$

This is equivalent to  $\mathcal{A}_{\mu+1}(q) = 0$ . So,  $q_{r_\mu r_\mu + 2}$  is vertical if and only if  $q_{r_\mu + 2}$  is vertical. So finally the letter of rank 2 in the in the **RVT** code of  $q_{r_\mu s_\mu}$  is  $V$  if and only if, the letters of rank

$$s_\mu + 1 = h_0 + h_1 + l_1 + 1 + \dots + h_\mu + l_\mu + 2$$

is also  $V$ .

Consider now an integer  $r_{\mu+2+l}$  with  $0 \leq l \leq l_\mu + h_\mu$ . Either  $q_{r_\mu r_\mu + l + 2}$  is critical or regular. If  $q_{r_\mu r_\mu + l + 2}$  was vertical, then, from the previous argument,  $q_{r_\mu + l + 2}$  must be also tangency which is a contradiction of the fixed **RVT** code. Assume that  $q_{r_\mu r_\mu + l + 2}$  is tangency. Then  $q_{r_\mu r_\mu + l' + 2}$  is also tangency for all  $0 \leq l' \leq l$  from (34) is tangency if and only if we have:

$$< x_{\mu+l'+3} - x_{\mu+l'+2}, x_{\mu+l'+2} - x_\mu > = 0 \text{ for all } 0 \leq l' \leq l$$

According to our assumption and the equations of  $\mathcal{C}_{R^{h_0}VT^{l_1}R^{h_1} \dots VT^{l_{\mu-1}}R^{h_{\mu-1}}}$  ( see proof of Theorem 5.2 property (ii)), it follows that  $q_{r_\mu r_\mu + l + 2}$  is tangency if and only if  $q_{r_\mu r_\mu + l + 2}$  is also tangency. .

On the other hand, by same argument,  $q_{r_\mu r_\mu + l + 2}$  is regular if and only if

$$\text{there exists } 0 \leq j \leq l \text{ such that } < x_{\mu+j+3} - x_{\mu+j+2}, x_{\mu+j+2} - x_\mu > \neq 0$$

Moreover, in this case,  $q_{r_\mu r_\mu + l' + 2}$  must be also regular for  $l \leq l' \leq l_\mu + h_\mu$

So, again analogue arguments used in the "tangency" case,  $q_{r_\mu r_\mu + l + 2}$  is regular if and only if  $q_{r_\mu + l + 2}$  is also regular.

It follows that that our assumption is then true for the integer  $\mu$ . □

*Proof of Theorem 5.2.*

According to Lemma 5.2 and Theorem 4.3 property (ii), to each induced articulated arm associated to a pair  $(r_i, s_i)$  there exists a family of directions  $\{K_0^i, \dots, K_{\kappa_i}^i\}$  of directions in  $\mathbb{R}^{m+1}$  such that  $q_{r_i s_i}$  belongs to the class  $\mathcal{C}_{RVT^{l_i}R^{h_i}} \subset \mathcal{C}^{r_i s_i}(m)$  if and only if in the configuration  $q_{r_i s_i}$  we have :

- each segment  $[M_{r_i+1+l}, M_{r_i+2+l}]$  is orthogonal in  $M_{r_i+1+l}$  to  $K_l^i$ , for  $l = 0, \dots, l_i$  ;
- each pair of consecutive segments  $[M_{l-2}, M_{l-1}]$  and  $[M_{l-1}, M_l]$  are not orthogonal in  $M_{l-1}$  for all  $r_i + 1 + l_i < l \leq r_i + 1 + l_i + h_i = s_i - 1$ .

On the other hand, each index  $i_\lambda$  is equal to  $r_\lambda + 2$  for  $\lambda = 1, \dots, \nu$ . So we get the announced results □

#### 5.4. EKR classes of 2-depth , RVT codes and configurations of articulated arms for $k \leq 4$ .

The combination of all possible **RVT** codes of depth 2 has an exponential growth relatively to the length  $k$  of special multi-flag. So, in this subsection we only describe the relations between EKR classes of 2-depth, **RVT** codes and configurations of articulated arms for  $k = 4$ . In fact this situation recovers the results of [4], [5] and [9].

At first for  $k = 3$  we have only  $\Sigma_{123}$  which is a EKR class of depth 2 and  $k = 4$  we have 14 EKR classes (of depth 2) whose numerical code are (see for instance [9]):

1111, 1112, 1121, 1122, 1123, 1211, 1212, 1213, 1221, 1222, 1223, 1231, 1232, 1233.

So, for  $k \leq 4$ , the other EKR classes are of depth 1.

On the other, at the end of section 4.3, we have seen that for  $k = 3$ , we have only one **RVT** class of depth 2 (i. e  $RT_0T_{01}$ ) but we have that we have 10 **RVT** classes for  $k = 4$  All other **RVT** classes are of depth 1.

For the decomposition of EKR classes of depth 1 into **RVT** classes are of depth 1 can be found in Theorem 5.2 and the corresponding interpretation in terms of configurations of an articulated arm can be found in Theorem 5.2. So we have only to give such result for EKR classes of depth 2 previously enumerated.

For this purpose we need the following Proposition:

**Proposition 5.1.**

Let be  $\Sigma_{j_1 \dots j_k}$  a EKR class of depth 2. Consider an index  $j_l = 3$  for  $l \geq 3$  and denote by  $\{i_1, \dots, i_\nu\}$  the set of index such that  $\{i : j_i = 2, 1 < i < l\}$

- (1)  $q \in \mathcal{C}^k(m)$  belongs to  $\Sigma_{j_1 \dots j_k}$ ,
- (2) the projection  $q_l = (x_0, \dots, x_l)$  of  $q$  onto  $\mathcal{C}^l(m)$  satisfies the following equations :

$$\begin{cases} \mathcal{A}_{l-1}(q_l) = 0 \\ \langle x_l - x_{l-1}, x_{l-1} - x_{i_{\lambda-2}} \rangle = 0 \text{ for some } 1 \leq \lambda \leq \nu \end{cases}$$

- (3) the articulated arm  $(M_0, \dots, M_k)$  satisfies the following situation at the configuration  $q$ :
  - the segments  $[M_{l-2}, M_{l-1}]$  and  $[M_{l-1}, M_l]$  are orthogonal in  $M_{l-1}$ ;
  - the segment  $[M_{l-1}, M_l]$  is orthogonal in  $M_{l-1}$  at the direction generated by the vector  $\overrightarrow{M_{i_{\lambda-2}} M_{l-1}}$  for some  $1 \leq \lambda \leq \nu$ .

For  $k = 4$ , we get easily the decomposition of EKR class of at most depth 2 into **RVT** classes as given in the following table and the proof is left to the reader. For  $k \leq 3$  the results are essentially particular cases of Theorem 5.2.

Decomposition of EKR classes into **RVT** classes

EKR class	<b>RVT</b> class
1111	$RRRR$
1112	$RRRV$
1121	$RRVR, RRV T$
1122	$RRVV$
1123	$RRT_0 T_{01}$
1211	$RVRR, RVTR, RVTT$
1212	$RVRV, RVT V$
1213	$RVRT_{01}, RVTT_{01}$
1221	$RVVR, RVVT$
1222	$RVVV$
1223	$RV T_0 T_{01}$
1231	$RT_0 T_{01} R, RT_0 T_{01} T_1, RV T_0 T_{01} T_2, RV T_0 T_{01} T_{12}$
1232	$RT_0 T_{01} V$
1233	$RT_0 T_{01} T_{01}, RT_0 T_{01} T_{02}$

For  $k \leq 4$ , we only give an interpretation of **RVT** classes in terms of configurations of articulated arm when the **RVT** code contains a letter of type  $T_{ij}$ . The other cases are particular cases of Theorem 5.2. The following results are essentially due to the Proposition 5.1 and the proofs are left to the reader.

Let be  $q = (x_0, \dots, x_k)$  a configuration of an articulated arm  $(M_0, \dots, M_k)$  with  $k \leq 4$ . We have the following characterizations:

•  $q$  belongs to  $\Sigma_{123} = \mathcal{C}_{RRT_0 T_{01}}$  if and only if  $[M_{i-2}, M_{i-1}]$  and  $[M_{i-1}, M_i]$  are orthogonal in  $M_{i-1}$  at  $q$  for  $i = 2, 3$  and  $[M_2, M_3]$  is also orthogonal at  $q$  to the direction generated by  $x_0 - x_2$  and no other orthogonality.

•  $q$  belongs to  $\Sigma_{1123} = \mathcal{C}_{RRRT_0 T_{01}}$  if and only if  $[M_{i-2}, M_{i-1}]$  and  $[M_{i-1}, M_i]$  are orthogonal in  $M_{i-1}$  at  $q$  for  $i = 3, 4$  and  $[M_3, M_4]$  is also orthogonal at  $q$  to the direction generated by  $x_1 - x_3$  and no other orthogonality.

•  $q$  belongs to  $\mathcal{C}_{RVRT_{01}} \subset \Sigma_{1213}$  if and only if  $[M_{i-2}, M_{i-1}]$  and  $[M_{i-1}, M_i]$  are orthogonal in  $M_{i-1}$  at  $q$  for  $i = 2, 4$  and  $[M_3, M_4]$  is also orthogonal at  $q$  to the direction generated by  $x_1 - x_3$  and no other orthogonality. The point  $q$  belongs to  $\mathcal{C}_{RVTT_{01}} \subset \Sigma_{1213}$  if and only if we have the previous constraints and moreover  $[M_2, M_3]$  is also orthogonal at  $q$  to the direction generated by  $x_0 - x_2$  and no other orthogonality.

•  $q$  belongs to  $\Sigma_{1223} = \mathcal{C}_{RV T_0 T_{01}}$  if and only if  $[M_{i-2}, M_{i-1}]$  and  $[M_{i-1}, M_i]$  are orthogonal in  $M_{i-1}$  at  $q$  for  $i = 2, 3, 4$  and  $[M_3, M_4]$  is also orthogonal at  $q$  to the direction generated by  $x_0 - x_3$  and no other

orthogonality.

• in  $\Sigma_{1231}$  we have :

- (i)  $q$  belongs to  $\mathcal{C}_{RT_0T_{01}R}$  if and only if  $[M_{i-2}, M_{i-1}]$  and  $[M_{i-1}, M_i]$  are orthogonal in  $M_{i-1}$  at  $q$  for  $i = 2, 3$  and  $[M_2, M_3]$  is also orthogonal at  $q$  to the direction generated by  $x_0 - x_2$  and no other orthogonality.
- (ii) The point  $q$  belongs to  $\mathcal{C}_{RT_0T_{01}T_1}$  (resp.  $\mathcal{C}_{RT_0T_{01}T_2}$ ) if and only if we have the previous constraints and moreover  $[M_3, M_4]$  is also orthogonal at  $q$  to the direction generated by  $x_0 - x_3$  (resp.  $x_0 - x_3$ ) and no other orthogonality.
- (iii) The point  $q$  belongs to  $\mathcal{C}_{RT_0T_{01}T_{12}}$  if and only if we have the previous constraints of (ii) and moreover  $[M_3, M_4]$  is also orthogonal at  $q$  to the directions generated by  $x_0 - x_3$  and by  $x_0 - x_3$  and no other orthogonality.

•  $q$  belongs to  $\Sigma_{1232} = \mathcal{C}_{RT_0T_{01}V}$  if and only if  $[M_{i-2}, M_{i-1}]$  and  $[M_{i-1}, M_i]$  are orthogonal in  $M_{i-1}$  at  $q$  for  $i = 2, 3, 4$  and  $[M_3, M_4]$  is also orthogonal at  $q$  to the directions generated  $x_0 - x_3$  and by  $x_0 - x_3$  and no other orthogonality.

•  $q$  belongs to  $\mathcal{C}_{RT_0T_{01}T_{01}} \subset \Sigma_{1233}$  if and only if  $[M_{i-2}, M_{i-1}]$  and  $[M_{i-1}, M_i]$  are orthogonal in  $M_{i-1}$  at  $q$  for  $i = 2, 3, 4$  and  $[M_2M_3]$  and  $[M_3, M_4]$  is also orthogonal at  $q$  to the direction generated by  $x_0 - x_2$  and  $x_0 - x_3$  respectively and no other orthogonality.

The point  $q$  belongs to  $\mathcal{C}_{RT_0T_{01}T_{02}} \subset \Sigma_{1213}$  if and only if  $[M_{i-2}, M_{i-1}]$  and  $[M_{i-1}, M_i]$  are orthogonal in  $M_{i-1}$  at  $q$  for  $i = 2, 3, 4$  and  $[M_2M_3]$  and  $[M_3, M_4]$  is also orthogonal at  $q$  to the direction generated by  $x_0 - x_2$  and  $x_1 - x_3$  respectively and no other orthogonality.

*Proof of Proposition 5.1.* According to Lemma 3.2 , at a point  $q_{l-1}$  the distribution  $\mathcal{D}_{l-1}$  is generated by the family:

$$\left\{ \sum_{r=1}^{m+1} (x_{l-1}^r - x_{l-2}^r) X_{l-1} + \Pi_{l-1} \left( \frac{\partial}{\partial x_{l-1}^r} \right) \right\}$$

Now, according the proof of Proposition 3.1 the distribution  $\mathcal{D}_l$  (which can be identified with  $(\mathcal{D}_{l-1})^{[1]}$ ) is generated by

$$\left\{ \mathcal{A}_{l-1} X_{l-1} + \sum_{r=1}^{m+1} (x_l^r - x_{l-1}^r) \Pi_{l-1} \left( \frac{\partial}{\partial x_{l-1}^r} \right), \Pi_l \left( \frac{\partial}{\partial x_l^r} \right) \right\},$$

According to Lemma 4.3 and the "operation"  $\mathbf{j}$  in the EKR classes, if  $q$  belongs to  $\Sigma_{j_1 \dots j_k}$ , we must have  $\mathcal{A}_{l-1}(q) = 0$ . So at such a point  $(\mathcal{D}_{l-1})^{[1]}$  is generated by

$$\left\{ \Pi_{l-1}(\mathcal{Z}_{l-1}), \Pi_l \left( \frac{\partial}{\partial x_l^r} \right) \right\},$$

But we have

$$\Pi_{l-1}(\mathcal{Z}_{l-1}) = \sum_{r=1}^{m+1} (x_l^r - x_{l-1}^r) \Pi_{l-1} \left( \frac{\partial}{\partial x_{l-1}^r} \right)$$

So by the same argument as before, if  $q$  belongs to  $\Sigma_{j_1 \dots j_k}$ , then there exists a direction  $v_q$  in the fiber over  $q_{l-1}$  such that  $x_l - x_{l-1}$  is orthogonal to  $v_q$ . In particular,  $z_l = x_l - x_{l-1}$  is a critical vector in the tangent space of the fiber over  $q_{l-1}$ . So, according to Remark 4.1 part (2) and Remark 4.6, the direction  $v_q$  must be equal to some  $x_{l-1} - x_{i_{\lambda}-2}$  for some  $1 \leq \lambda \leq \nu$ .

Clearly the converse is true. We get the equivalence (1)  $\Leftrightarrow$  (2)

The equivalence (2)  $\Leftrightarrow$  (3) is a consequence of Proposition 4.1 and Remark 4.6

□

## 6. CONCLUSIONS AND COMMENTARIES

According to the results of [7], [8], [4], [5] and [9], the results of this paper gives a decomposition of EKR classes in terms of **RVT** classes. However, this decomposition can be refined: for instance each **RVT** classes of code  $RVT$ ,  $RRVT$  and  $RVRV$  consist of two distinct "equivalence classes" (under the action of the group of (local) diffeomorphisms) (see [4] or [5]). On the other hand, in [9], the authors introduce the property of strong nilpotency. Again for instance the EKR class  $\Sigma_{121}$  splits into three "equivalence classes" while  $\Sigma_{121}$  is the union of only two **RVT** classes. It would be interesting to find some interpretation of this property in terms of configurations of an articulated arm.

On the other hand, as in [11] for  $k = 1$ , in [4] for  $k \geq 1$ , Castro and Montgomery look for the definition of **RVT** code in terms of germs curves at  $0 \in \mathbb{R}^{m+1}$  whose all directional derivatives at order  $k$  is a point of  $P^k(m)$ . They used the classification of curve singularities, under "Right-Left" equivalence (see Definition 3.11 in [4]), due to Arnold ([1]), to obtain beautiful results about classification of **RVT** classes in terms of Arnold's classification. Also let us mention [5] for complete results of classification of **RVT** classes in  $P^4(2)$ .

In this paper this aspect is not approached. However, it would be interesting to give an interpretation of a such approach in terms of germs of trajectories of the motion of an articulated arm.

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